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# The Philosophical Significance of Unitarily Inequivalent Representations in Quantum Field Theory

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# The Philosophical Significance
# of
# Unitarily Inequivalent Representations in Quantum Field Theory

**by**

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## Dedication

For Mom, Dad, Crissavol, Rok, Suzy, Bufford, Mac, Guzy, Thor,

and my family and friends…

## Acknowledgements

The journey of this dissertation has been possible due to the support, generosity, and love of many people. When I began thinking about writing a dissertation in the philosophy of physics, I did not want to write yet another tome on an interpretation of quantum mechanics or Bell's inequalities. I wanted a fresh new topic. In an advanced topics course in the philosophy of quantum theory, Fred Kronz introduced me to the algebraic approach to quantum field theory. Harald Atmanspacher gave some guest lectures on the algebraic approach, the puzzle of unitarily inequivalent representations, and Haag's theorem. I was intrigued. After class I was talking with Harald and Fred on the steps of Waggener hall about these topics, I asked Harald if he could recommend a good book to start learning the algebraic approach to quantum field theory. He thought for a moment and then told me that there were no good introductory books! The results were scattered through various papers and books. He then looked at me with a mischievous look and said that this would be a good dissertation project. I was in. Of course, this dissertation has expanded beyond the scope of what Harald suggested, but I hope that what you have before you provides a gentle introduction to the algebraic approach to quantum field theory, Haag's theorem, and the appearance of unitarily inequivalent representations in both algebraic quantum field theory and canonical quantum field theory.

During the writing process, I have worked at the Applied Research Laboratories (ARL) under Michael Revesz, who has been the most understanding supervisor one could ask for. Mike patiently listened to my mathematical questions and discussed the various stresses that accompany working in the academic world. I would also like to gratefully thank Michael Pestorius who gave me a personal research account and funding for the many conferences I have attended. Billy Ferguson has been a constant source of support, humor, and fellow longhorn sports enthusiast. I also gratefully acknowledge many other people who helped me in numerous ways at ARL including Karen Rhodes, Jan Focke, Marlana Rodgers, Christy Haebecker, and David Wick who helped me fix numerous formatting problems in this dissertation. Finally, I would like to gratefully thank and acknowledge Carole Freeman who tirelessly tracked down articles, books, conference proceedings, lectures, and every obscure incomplete citation I sent her. Her help has been crucial in the writing of this dissertation and incredibly generous.

My philosophical debts are numerous. I have benefited greatly from many philosophical discussions with Harvey Brown, Simon Saunders, Wayne Myrvold, John Earman, Laura Ruetsche, John Norton, and Nick Huggett. I especially would like to thank Harvey Brown who welcomed me into his home during my time at Oxford and for his friendship and advice. My philosophical education at the University of Texas at Austin has been advanced through the superb classes – both graduate and undergraduate – and discussions with Edwin Allaire, Ignazio

Angelelli, Nicholas Asher, Daniel Bonevac, Josh Dever, Jim Hankinson, Herbert Hochberg, Cory Juhl, Robert Kane, Louis Mackey, Kelly Oliver, Mark Sainsbury, and Johanna Seibt. My interdisciplinary educational trajectory has required numerous petitions that required arcane knowledge of various university policies and procedures. Fortunately, Jill Glenn always had an answer. Her extraordinary patience, compassion, and friendship have been a balm for so many graduate students that I have no idea how future graduate students will navigate through their graduate studies when she retires in a couple of months. I have also benefited from my interactions with many fellow graduate students including Dave Baker, Ari Duwell, Doreen Fraser, Sona Ghosh, Giovanni Valente, and Amy McLaughlin. Dave Baker always kindly provided me with a couch to crash on when I visited Princeton as well as feedback on my dissertation and greatly appreciated comradeship.

I would like to extend my gratitude and sympathy to members of my dissertation committee: Fred Kronz, Hans Halvorson, Cory Juhl, Josh Dever, and Larry Sklar. This dissertation is long and technically demanding in many places, but they always helped me focus on using my technical results in the service of my philosophical goals. I would like to highlight the contributions of three people in my philosophical development: Rob Clifton, Hans Halvorson, and Fred Kronz. I only had one year to work with Rob before he tragically passed away, but his enthusiasm and help were an inspiration to everyone he came in contact with. Every graduate student should strongly consider having Hans on his / her dissertation committee regardless of their topic.  Hans always gave me extensive comments on the rough drafts of my chapters and helped guide the development of my ideas.  It was Hans who suggested I call my interpretation of algebraic quantum field theory by the name of bidualism.  Fred has the distinction of winning the trifecta; he directed my senior honors thesis, my Master's thesis, and my dissertation.  While this strongly indicates the role he has played in my life, his friendship and support have provided the ying to my academic yang.  I might not have gone down this path in philosophy and decided to get degrees in mathematics and physics if not for Fred kindly listening to a young enthusiastic naïve undergraduate (is there any other kind?) when he showed up in his office wanting to work on quantum mechanics with no real training in either physics or mathematics.  The results of his efforts can be read off my CV: six degrees (B.A. degree in philosophy, B.S. degree in physics, B.S. degree in mathematics, M.A. degree in philosophy, M.A degree in the history and philosophy of science, PhD in philosophy).

As I was working on these acknowledgments, I regretfully learned of John Wheeler's passing on April 13, 2008.  I had the good fortune to meet him on several occasions.  Our first meeting would have a lasting impact on my desire to understand quantum mechanics.  After I decided to write a senior honors thesis in philosophy on quantum mechanics, I started reading the writings of Bohr and Wheeler.  I found out that John was going to be visiting the University of Texas, so I asked one of the physics secretaries if I could schedule an interview.  Fred

told me that Wheeler was always very busy when he visited so I should not expect him to have much time to visit with me. Upon entering the room, John greeted me warmly and immediately started asking me questions about philosophy. He asked me to explain Heidegger to him and recommend some introductory books on both continental and analytic philosophy. John thought that Heidegger's philosophical ideas might illuminate Bohr's doctrine of complementarity. We talked for nearly two hours about philosophy. Finally, John apologized for monopolizing the conversation and asked me what I would like to talk about. I pulled out a piece of paper on which I had typed a number of questions. After answering the first question, he asked for the piece of paper and proceeded to write down his answers. When he got to the third question he paused and thought for a minute. He then said, "This question gets to the heart of the matter about quantum theory." He told me to find the secretary and have her make a copy of my questions so he could look at them when he returned to Princeton. Needless to say I was jumping up and down inside. I floated down the hallway to carry out my mission. When I returned he kindly informed me that he would have to start working on his taxes which he had to get in the mail by midnight. As I got up to leave we both noticed that hanging on the wall was a picture of Einstein seated in his office in Princeton. John turned to me and said, "Talking with you about these issues reminds of the times I was discussing these very issues with Einstein in that room." You can imagine the effect these words had on me. That first meeting has always remained with me. I hope that this dissertation advances the discussion about the foundations of quantum theory.

I have a number of friends to thank who continued being my friend inspite of having no idea what I was working on except that it was “some kind of mathematical-quantum-physics-philosophy thing.” Laura Faulkner has been a pillar of strength for me during the writing of this dissertation. We were both working on our dissertations at the same time and decided to form a support group to help motivate us to keep writing even when we had some less than desirable rewriting task ahead of us. Each week we met and stated a goal that we would accomplish by the next week. It is surprising what an incentive it is to finish that task when you realize the weekly meeting day is approaching. Tina Radke has been a good friend through all of this. Kris Duda could easily fit in many of the sections above; he has read and commented on my papers and we have discussed many philosophical topics. When he finishes his dissertation, he will be one of the next great philosophers; no pressure Kris. I miss our marathon science fiction-martial arts movie sessions and your steak cooking skills. Amy McLaughlin and Dave Mattox have always included me in their life. Our nights in Austin eating Mexican food at Jorge’s and playing pool at Eric’s will always have a special place in my memories. You always listened to my sometimes insane ramblings and gave me sympathetic nods of the head. I dearly miss Apache. Sean Tierney has the distinction of having the longest tenure of FOT (Friends of Tracy). He has become an excellent emergency room doctor while overcoming

the childhood character flaw of being a Yankee / Mets / Knicks fan. As a young man he adapted and learned to be a fan of any team in first place as long as I am not a fan of said team. All kidding aside, if I ever need medical assistance, there is no one I would else I would so easily trust with my life.

My family is in some ways most responsible for my academic lifestyle. My aunt Denise, now Sister Ilia Delio, has not only encouraged my academic pursuit but been a role model for productivity with her many books and articles on numerous issues in theology. Unfortunately, none of my grandparents will be able to see the completion of my education, but I know that they would be very proud. I miss you Meme, Papa, Grandma, and Grandpa. My cousins and their beloved spouses have been fantastic and I always enjoy seeing them: Karen, Sharon, Vinny, Mark (I miss you), and David. A special acknowledgment goes to David who reinforced my love of philosophy and music with conversations that lasted from the late evening until early morning when we would each start falling asleep until one of us would say, “yeah, but what about…” I would also like to thank Gloria and Roger Kirby for their support of both myself and Crissavol. A special thanks goes to my English bulldogs (Suzy, Mac, Guzy, and Thor) who have slept at my feet while I sat at the computer typing away. My mother and father have had a profound influence on me. I would probably not be very successful at living outside of my head if not for my mother’s grounding influence. I have never met anyone so happy and connected to everyone around her. Her energy and pluck after working the night shift as a nurse and taking care of her son is unbelievable. I doubt I have ever had that much energy at any time in my life much less be able to do it on a day-to-day basis. My father has had more influence on me than anyone else in my life. He nurtured and refined my nature as a free thinker; encouraging me to question and reject what others told me if I had good reasons for doing so. Last but certainly not least, I would like to thank my wife Crissavol for her love and support.

**The Philosophical Significance**

**of**

**Unitarily Inequivalent Representations in Quantum Field Theory**

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Supervisor: Frederick Kronz

This dissertation gives a general account of the properties of unitarily inequivalent representations (UIRs) in both canonical quantum field theory and algebraic quantum field theory. A simple model is constructed and then used to show how to build a broad spectrum of UIRs including a version of Haag's theorem. Haag and Kastler, two of the founding fathers of algebraic quantum field theory, argue that the problems posed by UIRs are solved by adopting a notion of equivalence that is weaker than unitary equivalence, which they refer to as physical equivalence. In the dissertation, it is shown that their notion does not provide a suitable classificatory schema. Some of the most important physical representations fail to satisfy the mathematical conditions of their notion. However, Haag and Kastler's notion has an unexpected connection with classical observables. A theorem is proven in which two representations make the same predictions with respect to all classical observables if and only if they satisfy their notion of physical equivalence. Following Haag and Kastler's lead, it was claimed by most proponents of algebraic quantum field theory that all physical content resides in a specific class of observables. It is shown in the dissertation that such claims are exaggerated and misleading. UIRs are used to elucidate the nature of quantum field theory by showing that UIRs have different expectation values for some classical observables of the system, such as temperature and chemical potential, which are not in Haag and Kastler's specific class. It is shown how UIRs may be used to construct classical observables. To capture the physical content of quantum field theory it is shown that a much larger algebra than that of Haag and Kastler is necessary. Finally, the arguments that UIRs are incommensurable theories are shown to be flawed. The lesson of UIRs is that the mathematical structures in both canonical quantum field theory and Haag and Kastler's version of algebraic quantum field theory are not sufficient to capture all of the physical content that UIRs represent. A suitable algebraic structure for quantum field theory is provided in the dissertation.

## Table of Contents

# List of Tables

# List of Figures

# Chapter One

## 1.1 Introduction

Quantum field theory effectively synthesizes three physical theories: quantum mechanics, special relativity, and classical field theory. Though quantum field theory (QFT) is our most empirically successful theory, its mathematical and conceptual basis has been troubled since its birth. This has lead to the development of numerous frameworks and techniques for calculating specific problems. Renormalization was a successful collection of mathematical techniques that was able to make predictions that were experimentally verified, but this did not really convince everyone that a suitable theory of quantum fields had been found.

> Renormalized quantum electrodynamics is by far the most successful theory we have today. This very impressive fact, however, does not make the whole situation less strange. We start out from equations which do not make sense. We apply certain prescriptions to their solutions and end up with a power series of which we do not know that it makes sense. The first few terms of this series, however, give the best predictions we know. Things do not become more understandable by the fact that the success of quantum electrodynamics is completely singular…The most impressive feature of the development of quantum electrodynamics is the astonishing ability of the theory to survive. Exactly from what this faculty derives is unknown. The analysis of other models can hardly shed light on this circle of problems. (Jost 1965, xiii)

Most physicists would no longer agree with Jost. The conventional wisdom is that these problems have been solved. However, the foundational issues that Jost refers to remain.

> Another conceptual reason for a reexamination of the foundational problems at the origin of QFT is that after more than seven decades no non trivial (even non realistic) model in four (space time) dimensions is under non perturbative control. Actually, the prototypic model of self interacting scalar field, which is used in most textbooks for developing (non trivial) perturbation theory, has been proved to be trivial (namely the renormalized coupling constant vanishes when the ultraviolet cutoff is removed) under general conditions, when treated non perturbatively. This means that in general the perturbative expansion is not reliable and in general one cannot define a QFT model by its perturbative expansion.
> It should be stressed that most of our wisdom in QFT is derived from the perturbative expansion and that it would be silly to neglect the extraordinary success of perturbative quantum electrodynamics (QED) in yielding theoretical predictions which agree with the experiments up to the eleventh significant figure. On the other hand, soon after the setting of perturbation theory, Dyson argued that the perturbative expansion of QED cannot be summed and that big oscillations overwhelming the so successful lowest orders are expected to arise…These negative results legitimate the need of a non-perturbative approach to the problem of combining quantum mechanics and relativity, with the aim of either validating the foundations of quantum field theory or displaying the need of radical changes and new ideas. (Strocchi 2004, 501-502)

In this dissertation, I focus on one crucial aspect of the problems alluded to by Strocchi in the case of modeling interactions in QFT: the appearance of unitarily inequivalent representations (UIRs). By examining this issue, I will develop an alternative mathematical approach to quantum field theory.

The topic of UIRs in QFT has been the subject of many recent philosophy of physics papers. A proper understanding of UIRs is crucial for the philosophical foundations of QFT since (1) they have been characterized as incommensurable physical theories, (2) their properties have been used to argue that particles do not exist, (3) they are connected to problems in modeling dynamics, which is related to the triviality problem, and (4) they have been used to show how apparently different theories are equivalent.

One of the first significant applications of unitarily *equivalent* representations occurred during the birth of quantum mechanics. During the mid-to-late 1920's, there were two competing versions of non-relativistic, spinless quantum mechanics: Schrödinger's wave mechanics and Heisenberg's matrix mechanics. Wave mechanics and matrix mechanics were considered rival quantum theories. Schrödinger and Einstein favored wave mechanics, while Heisenberg championed his matrix mechanics. Neither camp thought very highly of the competing theory.

> I was discouraged, if not repelled, by what appeared to me as very difficult methods of transcendental algebra, defying any visualization.
> (Schrödinger quoted in (Mehra and Rechenberg 2000, 42))

> I am convinced that you have made a decisive advance with your formulation of the quantum condition, just as I am equally convinced that the Heisenberg-Born route is off the track.
> (Letter from Einstein to Schrödinger April 26, 1926 quoted in (Mehra and Rechenberg 1987, 626))

> The more I think of the physical part of the Schrödinger theory, the more detestable I find it. What Schrödinger writes about visualization makes scarcely any sense, in other words I think it is shit. The greatest result of his theory is the calculation of matrix elements.
> (Letter from Heisenberg to Pauli June 6, 1926 quoted in (Styer 2000, 127))

Through the work of Schrödinger, Dirac, Stone, and most importantly von Neumann, it was realized that the two theories are unitarily equivalent. Roughly, this equivalence may be expressed by saying that each of them is a representation of the Heisenberg form of the canonical commutation relations (CCRs) for the momentum operator $P$ and the position operator $Q$:[1]

$$\left[P_i, Q_j\right] = P_i Q_j - Q_j P_i = -i\hbar\delta_{ij}$$
$$\left[P_i, P_j\right] = \left[Q_i, Q_j\right] = 0$$

The indices $i$, $j$ correspond to degrees of freedom of a system. In Schrödinger's representation, $P$ corresponds to the differential operator $-i\hbar\dfrac{d}{dx}$ and $Q$ to multiplication by the variable $x$. In Heisenberg's matrix representation, Born and Jordan found formal matrices with infinitely many entries to represent the operators $P$ and $Q$.

Using the operators associated with $P$ and $Q$ above is complicated by the fact that these operators are unbounded. As a result, they are only defined on a merely dense subset of the Hilbert space. In other words, they are only defined on some but not all of the vectors in a Hilbert space. Until this subset of vectors has been specified, these operators are essentially ambiguous. To circumvent the problems associated with unbounded operators, it is standard practice to work with unitary operators created by exponentiating separately $P$ and $Q$. Unitary operators are bounded, so they are defined on the entire Hilbert space. When unitary operators act on vectors they do not change their length. The associated unitary operators, which are called the Weyl operators, are defined as:

[1] The original Stone-von Neumann proof of the unitary equivalence of wave mechanics and matrix mechanics uses the Weyl form of the CCRs instead of the Heisenberg form of the CCRs.

$$U(a_i) = e^{\frac{-i\, a_i P_i}{\hbar}} \text{ and } V(b_j) = e^{\frac{-i\, b_j Q_j}{\hbar}},$$

for any $a_i, b_j \in \mathbb{R}$. The CCRs for the Weyl operators, which is called the Weyl form of the CCRs, is defined as follows.

$$U(a_i)V(b_j) = e^{\frac{-i\, a_i b_j}{\hbar}} V(b_j)U(a_i)$$

The Weyl operators act on elements of $L^2(\mathbb{R})$, the space of Lebesgue square integrable functions on the set of real numbers, in the following way:

$$U(a_i)\,\Psi(x_i) = \Psi(x_i - a_i) \text{ and } V(b_j)\,\Psi(x_j) = e^{\frac{-i\, b_j x_j}{\hbar}}\Psi(x_j),$$

where $\Psi(x_i), \Psi(x_j) \in L^2(\mathbb{R})$.

In 1931, von Neumann proved an important theorem that is now referred to as the Stone-von Neumann theorem. The following is a contemporary formulation of the Stone-von Neumann theorem abstracted from (Summers 2001).

**Stone-von Neumann Theorem:** If the functions $a_i \to \tilde{U}(a_i)$ and $b_j \to \tilde{V}(b_j)$ are weakly continuous and $\{\tilde{U}(a_i) | a_i \in \mathbb{R}\}$, $\{\tilde{V}(b_j) | b_j \in \mathbb{R}\}$ are unitary operators acting irreducibly on a separable Hilbert space $\mathcal{H}$ where $1 \le i \le n$, $1 \le j \le n$, $\tilde{U}(a_i)\tilde{V}(b_j) = e^{\frac{-i\, a_i b_j}{\hbar}} \tilde{V}(b_j)\tilde{U}(a_i)$, $\tilde{U}(a_i)\tilde{U}(b_j) = \tilde{U}(a_i + b_j)$, and $\tilde{V}(a_i)\tilde{V}(b_j) = \tilde{V}(a_i + b_j)$,then there is a Hilbert space isomorphism

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The Weyl form of the CCRs is discussed below.

$W : \mathcal{H} \to L^2(\mathbb{R})$ such that $W\tilde{U}(a_i)W^{-1} = U(a_i)$ and $W\tilde{V}(a_i)W^{-1} = V(a_i)$ for all $a_i \in \mathbb{R}$.[2]

The Stone-von Neumann theorem makes four key assumptions: (1) it assumes that the representation is irreducible, (2) the operators can be exponentiated and satisfy the Weyl form of the CCRs, (3) the unitary operators satisfy weak continuity, and (4) it only holds for systems with a finite number of degrees of freedom represented by the indices *i* and *j*. When the Stone-von Neumann theorem fails to hold (i.e., one of these four conditions fails to hold), UIRs are mathematically possible. It is sometimes thought that UIRs only occur when the systems under consideration have an infinite number of degrees of freedom. However, Schmüdgen (1983) proved that there are an uncountable number of representations of the Heisenberg form of the CCRs that are unitarily inequivalent to the Schrödinger representation for systems with a finite number of degrees of freedom. When these operators are exponentiated they do not satisfy the Weyl form of the CCRs.

There are two important consequences of the Stone-von Neumann Theorem. First, the physical content of any irreducible representation of $\{\tilde{U}, \tilde{V}, \mathcal{H}\}$ is the same. In particular, wave mechanics and matrix mechanics have the same physical content and produce the same predictions for physical

[2] *Weak continuity* of $U(t)$ with respect to the parameter *t* means that $\langle \Phi | U(t) | \Psi \rangle$ is a continuous function of *t* for each $\Phi, \Psi \in \mathcal{H}$.

models. The simplest way to see their empirical equivalence is through their unitary equivalence. In quantum mechanics, states and observables correspond to special types of linear operators; states correspond to density operators and observables to self-adjoint operators. The expectation value for an observable A for a system that is in the state $\rho$ is $Tr\left(\rho A\right)$.[3] The transformation of an operator $O_1$ from one representation to a corresponding operator $O_2$ in the other is effected by a unitary transformation *U* as $O_2 = UO_1U^{-1}$. Let $\rho_1$ be a state and $A_1$ be an observable in one representation, and let $\rho_2$ and $A_2$ be their counterparts in the other representation obtained via this unitary transformation. This unitary transformation preserves expectation values; i.e.,

$$\begin{aligned} Tr\left(\rho_2 A_2\right) &= Tr\left(U\rho_1 U^{-1} U A_1 U^{-1}\right) = Tr\left(U\rho_1 I A_1 U^{-1}\right) \\ &= Tr\left(U\rho_1 A_1 U^{-1}\right) = Tr\left(U^{-1} U\rho_1 A_1\right) = Tr\left(I\rho_1 A_1\right) \\ &= Tr\left(\rho_1 A_1\right) \end{aligned}$$

since the trace operation is invariant under cyclic permutations and $U^{-1}U = I = UU^{-1}$, where *I* is the identity operator. The empirical equivalence follows from unitary equivalence since the equality above holds for all possible states and observables; the associated set of expectation values exhausts the physical content of the theory. What this means in practice is that one may freely choose to work with the most convenient representation. If a particular problem

[3] Given an orthonormal basis $\left|e_i\right\rangle$ for a Hilbert space $\mathcal{H}$, the trace of an operator *B* is defined as $Tr\ B = \sum_{i=1}^{N} \left\langle e_i \middle| B \middle| e_i \right\rangle$.

is more easily solved in matrix mechanics than wave mechanics, the unitary equivalence of the two representations guarantees that the empirical predictions would be exactly the same if they were instead solved in wave mechanics. Unitary equivalence can be thought of roughly as three requirements:

(1) There exists an isomorphism[4] between the sets of states of two theories.

(2) There exists an isomorphism between the sets of observables of two theories.

(3) The theories make the same predictions; they are empirically equivalent.

The Stone-von Neumann theorem does not apply to QFT because quantum fields are defined on each spacetime point and thus have an infinite number of degrees of freedom.[5]

The equivalence of wave mechanics and matrix mechanics is an example of what Sklar (2000, 14) calls an argument for ontological elimination. In this case, the Stone-von Neumann theorem showed that neither the ontology associated with wave mechanics nor the ontological claims associated with matrix mechanics had a claim as the unique ontology of quantum mechanics. Since the Stone-von Neumann theorem is not applicable in QFT, is there a critical reconstruction of QFT that shows that UIRs are mathematical artifacts which are not part of the real content of QFT? Sklar suggests that UIRs can be ontologically eliminated in QFT. "[I]n the quantum-field-theoretic case there are

[4] A bit more precisely, the isomorphism preserves the structure of the inner product which secures the empirical equivalence of the theories.

too many unitarily inequivalent but observationally equivalent representations available in the standard axiomatic field theory." (Sklar 2000, 20) It would certainly be more convenient if some argument showed that all of these UIRs are in some sense observationally equivalent. The argument I believe Sklar has in mind was formulated in the algebraic approach by Haag and Kastler (1964). Even though the representations in QFT are usually not unitarily equivalent, they argued that they should be considered *physically equivalent*, which is usually taken to imply that they are observationally equivalent. This argument is critically examined in chapter four. Sklar (2000, 102) also suggests that one of the most well-known results connected with UIRs, Haag's theorem, might also be a mathematical artifact.

The conclusion of this dissertation is an anathema to the ontological elimination of UIRs that Sklar seems to be in favor of. This dissertation shows how UIRs are crucial for modeling many different types of physical systems not only in QFT but also in quantum statistical mechanics (QSM). UIRs typically have significant physical differences. By focusing on the arguments made in QFT for minimizing or eliminating UIRs, the mathematical and conceptual structure of QFT and QSM becomes more lucid. This dissertation articulates more general mathematical structures which show how UIRs fit together, can be compared, and can provide partial accounts of the world. Even though these conclusions are at odds with the kind of ontological elimination that Sklar

[5] However, quantum fields do not have a continuum of degrees of freedom. For quantum fields to be mathematically well-defined they must be "smeared out" with test functions. This reduces

discusses, this dissertation is an example of another type of philosophical critique within foundational science that Sklar (2000, 141) advocates: the search for a more general mathematical structure within which a theory can be reconstructed.

## 1.2 Outline of the Dissertation

Haag's theorem is a very important result in QFT. Intuitively, it is supposed to show that the mathematical structure associated with QFT[6] is incapable of modeling interactions. It has also been interpreted as showing that a representation of a free quantum field is unitarily inequivalent to a representation of an interacting quantum field. There are many different theorems and results that are referred to as "Haag's theorem" and so it has come to mean different things to different people. Chapter two examines the history of Haag's theorem and shows how Haag's work combines the earlier ideas of van Hove (1951) (1952) and Friedrichs (1953). It also discusses more general and mathematically rigorous proofs including those by Wightman and Hall, Greenberg, and Jost. Once these results and their interrelations are clarified it becomes easier to explain how these different conceptions of Haag's theorem are connected and how they relate to UIRs.

There are very few simple examples of UIRs in the physics literature that are examined in detail. Chapter three demonstrates one of the simplest methods

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the degrees of freedom to being countably infinite. See chapter three for more details.

for constructing UIRs using a shift model. This model and its variants are able to capture many different types of UIRs. Specifically, a simplified version of Haag's theorem can be proved which exhibits how the spatial invariance of the vacuum is sufficient to generate UIRs. It is also shown how the main features of the Unruh effect can be reproduced for any pair of UIRs in the model. Finally, it is shown how the results of the model hold not only at the creation and annihilation operator level, but at the quantum field level as well. At the quantum field level, a new property I call *hypermyriotic* is examined.[7]

The analysis of UIRs in chapter three is carried out in the canonical quantum field theory framework, which uses Fock spaces. While this approach to quantum field theory is still taught, it is not mathematically rigorous.[8] One of the most mathematically rigorous and philosophically interesting formulations of quantum field theory is called *algebraic quantum field theory* (AQFT). Unfortunately, the conceptual landscape of AQFT is difficult for many people to grasp. A self-contained introduction to the mathematics underlying AQFT is provided in the first section of chapter four. AQFT is especially crucial to understanding the physical and philosophical significance of UIRs since some of its proponents, such as Haag and Kastler, have an argument that claims that UIRs are physically *insignificant*. Though these representations fail to be unitarily

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[6] More specifically, the Fock representation in canonical QFT cannot be used to model interactions.

[7] The property of a representation being myriotic or amyriotic occurs at the creation / annihilation operator level. The term "hyper" in hypermyriotic refers to the occurrence of a property similar to the definition of a myriotic representation at the quantum field level.

equivalent to each other, these AQFT proponents claim that they can be considered *physically equivalent* to each other. This argument is critically examined and shown to fail in many physically significant cases. However, a theorem is proved which shows how the mathematical condition underlying their notion of physical equivalence can be understood as a condition for classical equivalence. Classical observables such as temperature and chemical potential are then constructed in a way that uses UIRs. Finally, these notions are examined in the context of the algebraic formulation of the Unruh effect.

The issue of UIRs in AQFT brings the question of how to interpret AQFT to the centerstage. Which mathematical structures in AQFT are capable of capturing its physical content? This is a debate that philosophers of physics have recently begun to explore. The conceptual options are discussed in detail in chapter five. The position of "bidualism" is defended as the best interpretation. The mathematical structure associated with it can capture the physical content of AQFT, algebraic quantum statistical mechanics (AQSM), as well as the physical differences between different UIRs. It has been argued by (Arageorgis, Earman, and Ruetsche 2002b) that different UIRs can be considered incommensurable physical theories and worlds that are impossible relative to each other. These arguments are examined and it is shown that UIRs do not possess any such radical notion of incommensurability. Conclusions are given in chapter six.

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[8] For example, for the creation and annihilation operators to be mathematically well-defined they need to be "smeared out" with test functions. See chapter three for more details.

Parts of this dissertation are based on some of my previously published articles. Chapter two was published in (Lupher 2005). A small part of chapter one and chapter four come from (Kronz and Lupher 2005).

# Chapter Two

## 2.1 INTRODUCTION

In the physics literature, there are several different characterizations of Haag's theorem and its consequences for QFT. These different versions of Haag's theorem are due in part to various generalizations and more "rigorous" proofs of Haag's theorem as well as to the fact that many of these proofs were done using different formulations of QFT. As a result, there is confusion about what Haag's theorem is and when it was proved. This chapter clears up some of this confusion by examining the history and development of Haag's theorem up to 1959. It is argued that the question of who proved Haag's theorem is tied up with what the theorem is taken to show.

Haag's original theorem assumed that there are two sets of field operators that satisfy the CCRs: (1) the free, or asymptotic fields, which occur at time $t = \pm\infty$, and (2) the "actual" fields, which occur at any finite time and that the theory can be formulated entirely from them. He also assumed that (3) there is a unique invariant normalizable vacuum state for the theory, that (4) there is a positive definite energy spectrum, and that (5) the "actual" fields are transformed by unitary operators representing translations in space. From these assumptions, Haag showed that (1) and (2) cannot belong to the same representation of the CCRs; they are UIRs.

Haag's theorem is an important result in QFT that has not been examined in much depth in the philosophy of physics literature. What it is and what follows from it has been an area of controversy in physics. Philosophers of physics have tended to view it as an important area for study, but few have done much to clarify the nature and scope of Haag's theorem. To the extent that philosophers of physics do discuss Haag's theorem, their analysis is promissory at best.[9]

## 2.2 What is Haag's Theorem?

Haag's theorem is generally taken to show that there are severe to insurmountable mathematical difficulties modeling interactions in QFT. The nature and extent of these mathematical difficulties are the loci of debate about the significance (or insignificance) of Haag's theorem. Here is how Teller, a philosopher of physics, described Haag's theorem:

> According to something called Haag's theorem there appears to be no known consistent mathematical formalism within which interacting quantum field theory can be expressed. (Teller 1995, 115)

A more conservative description is given by the physicists Streater and Wightman.

> Haag's theorem is very inconvenient; it means that the interaction picture exists only if there is no interaction.
> (Streater and Wightman 2000, 166)

[9] This part of the dissertation was written before the excellent article by Earman and Fraser (2006) was published.

Streater and Wightman do not make the stronger statement, as Teller did. Rather, they take Haag's theorem to show that the interaction picture is empty of interactions in canonical quantum field theory on Fock space.[10] But in either case Haag's theorem is a significant result for the foundations of QFT. What conclusions have philosophers of physics reached on Haag's theorem?

### 2.3 Conflicting Reactions of Philosophers of Physics

As shocking as Haag's theorem appears, philosophers of physics have done very little to explicate it. The few who mention it tend to regard it as something important that someone (else) should investigate thoroughly.

> [Haag's theorem] implies, for example, that the only QFTs that exist in the interaction picture describe free fields. Since this is the framework used by physicists to describe the interacting theories of nature, the theorem seemingly presents a paradox.
> (Huggett and Weingard 1996, 306)

> Everyone must agree that as a piece of mathematics Haag's theorem is a valid result that at least appears to call into question the mathematical foundations of interacting quantum field theory, and agree that at the same time the theory has proved astonishingly successful in application to experimental results. What seems less clear is how the assumptions of the theorem should be brought to bear on both the product and the interpretation of the theory…I have no light to throw on these important questions. In this chapter my exposition will proceed along lines almost universally accepted by practitioners of the theory, disregarding Haag's theorem. (Teller 1995, 115-116)

[10] Presumably, one reason they did not make the stronger statement put forth by Teller was the limited success of constructive field theory; the demonstrated ability to model specific interactions in two and three dimensions but not yet in four dimensions.

But there is also a dissenting view in the philosophy of physics literature about the significance of Haag's theorem.

> There may be a presence within a theory of conceptual problems that appear to be the result of mathematical artifacts. These seem to the theoretician to be not fundamental problems rooted in some deep physical mistake in the theory, but, rather, the consequence of some misfortune in the way in which the theory has been expressed. Haag's Theorem is, perhaps, a difficulty of this kind. (Sklar 2000, 102)

One thing to notice in these quotations is the use of modifiers such as "seemingly," "at least appears to," and "perhaps" in their discussion of Haag's theorem. No one has taken a very firm stand on what the consequences of Haag's theorem may be. Nor has anyone provided an argument for the significance or insignificance.[11] A plausible explanation for this hesitancy to take a stand on Haag's theorem is obtained by doing a quick search of the physics literature. One will find many different papers claiming to prove, "rigorously" prove, or to prove a "generalized" version of Haag's theorem. These proofs are sometimes done under different formulations of QFT such as Wightman's axiomatic approach or the LSZ approach. Thus, it is not clear whether Haag's theorem applies generally or only to some approaches to QFT and not to others.

[11] The only exception that I have found is an article by (Heathcote 1989), but his article is mainly focused on his view of causality.

## 2.4 Conflicting Reactions of Physicists

There are several different opinions about the significance of Haag's theorem in the physicist community. For example, Wightman and Roman considered Haag's theorem an important result in quantum field theory.

> [T]here is a widespread opinion that the phenomena associated with Haag's Theorem are somehow pathological and irrelevant for real physics…I make one more attempt to explain why that is not the case. (Wightman 1965a, 245)

> Haag's Theorem is very deep…The most sobering consequence of Haag's theorem is that the interaction picture of canonical field theory cannot exist unless there are no interactions. (Roman 1969, 391)

On the insignificance side of the debate is Källén who said the following.

> [T]he theorem discovered by van Hove and Friedrichs and usually referred to as the "Haag theorem" is really of a very trivial nature and it does not mean that the eigenvalues of a Hamiltonian never exist or anything that fundamental. (Källén 1962, 170)

The connection between Haag's theorem and certain problems with the Hamiltonian that Källén mentioned in the quotation above will be discussed below in connection with van Hove's work.

Other field theorists choose not to worry about Haag's theorem or its possible implications on their work. Their calculations have been empirically verified, and they have little concern about a mathematical result that says that they may not be calculating the results of various interactions.

> [L]et us first ask what we are to make of it when we find practicising field theorists plunging ahead, presenting their theory with blithe

> indifference to the problems posed by Haag's theorem. As I understand the history of the subject, quantum field theory was developed in ignorance of these mathematical problems. Indeed, the theory was initially formulated and applied with astonishing empirical success in the late 1940s, while the difficulties here in question did not come to light until the mid 1950s. Even the existence of the problems did not become widely known. And even when they were appreciated, most field theorists were not about to let such formal problems get them sidetracked from the obviously impressive successes of their theories. Work continued as if these formal problems did not exist – theorists took a "Damn the mathematical torpedoes, full speed ahead!" attitude. (Teller 1998, 157)

Some physicists who have heard of Haag's theorem misunderstand it. Teller (1998, 157) recalled one instance of talking to a "prominent field theorist" about Haag's theorem who incorrectly dismissed it as having to do with issues of mathematical rigor associated with the delta function. It is stories like these that most likely led to Streater (1975, 796) calling Haag's theorem, "one of the most widely misquoted results of [QFT]."[12]

### 2.5 The Neglect of Haag's Proof

If one looks for a discussion of Haag's theorem in books on QFT, one will hardly find a mention of it in textbooks written after the 1970's. The standard textbooks of QFT such as Peskin and Schroeder (1995), Ryder (1996), and

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12 Streater cited as an example of this the textbook by Bjorken and Drell (1965). In their chapter on perturbation theory, they assumed that the interacting and incoming free fields were connected at each time *t* by a unitary transformation. In a footnote on page 175, they stated that the existence of such a unitary transformation breaks down for systems with a nondenumerable number of degrees of freedom and cite Haag's 1955 paper, but then they assume the existence of such a unitary transformation!

Weinberg (1995) do not mention Haag's theorem. If one wants to find textbook discussions of Haag's theorem, then it is necessary to look at textbooks written in the 1960's and 1970's, e.g. Roman (1969), Barton (1963), Streater and Wightman (2000) (which was originally published in 1964), and Bogolubov (1975). However, these discussions of Haag's theorem focus on Wightman's proof of a "generalized" Haag's theorem and they do so in the context of Wightman's axiomatic formulation of QFT. There is rarely any analysis of Haag's original proof. This begs the question as to why these quantum field theorists do not bother to analyze Haag's original proof.

Part of the answer is that in the 1960's and 1970's quite a bit of work was being done on new approaches to QFT, while Haag's original paper was based on QFT from the late 1940's through the early 1950's. I have only found two sources that do more than merely cite Haag's paper. In Hall and Wightman's paper in which they proved their generalized Haag's theorem, they indicated that Haag's original proof was inconclusive

> In the opinion of the present authors, Haag's proof is, at least in part, inconclusive…It will not escape the discerning reader of Haag's paper that, while we have generalized his results, eliminated one of his assumptions (the asymptotic condition), completed his proofs, and sharpened his conclusions, the essential physical points are Haag's.(Hall and Wightman 1957, 41)

Unfortunately, they do not characterize the shortcomings of Haag's proof or why it was inconclusive. The other source is a review of Haag's paper by Dyson (1955), where he said that the "so-called Haag's Theorem…is essentially an old theorem of L. Van Hove but is here presented in much greater generality."

Dyson also took Haag to task somewhat for not providing any constructive solution to the problem of interactions. Though it is unclear why Dyson is so critical of Haag's paper, there is one reason to be suspicious. In the abstract of Haag's paper, he wrote, "It is shown that the "free field vacuum" of the Tamm-Dancoff method and Dyson's matrix $U(t_1, t_2)$ for finite $t_1$ or $t_2$ cannot exist" (Haag 1955b, 1).

### 2.6 INFLUENCES ON HAAG'S FORMULATION OF THE THEOREM

Dyson and Källén suggest (in the quotations above) that they felt that Haag's theorem was based on the work of van Hove and Friedrichs. From Haag's original paper (1955b) we know that he was familiar with the work of van Hove (1952) and Friedrich (1953) as well as a preprint of a paper by Wightman and Schweber (1955a). We also know that the ideas for Haag's paper were presented in lectures that he gave at the CERN theoretical study group from 1952 to 1953. I will give a brief review of these influences in this section.

#### 2.6.1 Van Hove's Work

There are two main papers of van Hove (1951) (1952) that are usually cited in connection with Haag's theorem and the mathematical problems involved in modeling interactions in quantum field theory. In his 1951 paper, van Hove investigated the mathematical properties of the interaction Hamiltonian $H_I$ and the total Hamiltonian $H = H_B + H_F + gH_I$, where $H_B$ is the free boson Hamiltonian, $H_F$

is the free fermion Hamiltonian, $H_I$ is the interaction Hamiltonian, and $g$ is a dimensionless coupling constant, in QFT.[13] He was interested in whether $H$ and $H_I$ exist as well-defined operators on the Hilbert space $S_o$ of the normalized stationary states $\varphi_\alpha$ of the free fields. To answer this question, van Hove assumed that the system is put in a cubic box with periodic boundary conditions on the walls, which was a typical assumption for the time (Wentzel 1949, 27). The periodic boundary conditions change the continuous space into a lattice. Within each box, the momentum of the total and interaction Hamiltonian is cut off for some value $K$ and the stationary states $\varphi_\alpha$ have finite energy and are characterized by having a finite number of particles (bosons, fermions, and anti-fermions) in specific plane wave states. The original Hamiltonians are recovered formally by $H = \lim_{K\to\infty} H^K$ and $H_I = \lim_{K\to\infty} H_I^K$. $S_o$ is the Hilbert space formed from linear combinations of the $\varphi_\alpha$ vectors: $\varphi = \sum_\alpha c_\alpha \varphi_\alpha$ with $\varphi_\alpha \in S_o$, $c_\alpha$ complex, and $\sum_\alpha |c_\alpha|^2 < \infty$. The domain of the free Hamiltonians for fermions and bosons is $S_o$. $H^K$ and $H_I^K$ are densely defined operators on $S_o$. The main result of this paper is that for any nonzero vector $\varphi = \sum_\alpha c_\alpha \varphi_\alpha$ in $S_o$, the total Hamiltonian and the interaction Hamiltonian cannot be defined on $S_o$ because they have infinite norms: $H\varphi = \sum_\alpha c_\alpha{}' \varphi_\alpha$ and $H_I\varphi = \sum_\alpha c_\alpha{}'' \varphi_\alpha$ with $\sum_\alpha \left|c_\alpha{}'\right|^2 = +\infty$ and $\sum_\alpha \left|c_\alpha{}''\right|^2 = +\infty$.

13 If $g$ = 0, there is no interaction and only free fields are present.

Even if $H^K\varphi$ and $H_I^K\varphi$ are vectors in $S_o$, when the cutoff is removed by taking the limit as $K$ goes to infinity the resulting operator yields a vector that is not in the Hilbert space of the free fields: $\lim_{K\to\infty}\left\|H^K\varphi\right\| = \infty = \lim_{K\to\infty}\left\|H_I^K\varphi\right\|$.[14]

While van Hove is generally credited as the first person to demonstrate some of the mathematical problems with treating the Hamiltonian as a well defined operator in QFT, there are two sources that he cited where these problems were discussed earlier. Snyder (1950) mentioned that the Hamiltonian when applied to a state vector maps that state vector into a vector of infinite length (1950, 520). Van Hove obtained some preliminary results for his 1951 paper in his collaboration with Gossiaux whose dissertation (1950) was on the domain of the Hamiltonian in QFT.

In both his 1951 and 1952 papers, van Hove believed that infinite tensor product spaces might be the appropriate mathematical structure to model interactions and on which the Hamiltonian is well defined. In his 1951 paper, he wanted to expand the space of stationary states to include not only states where there are a finite number of particles present, but also stationary states where there are an infinite number of particles present. There are a nondenumerable number of these states and they have infinite eigenvalues from $H_B + H_F$. This much larger Hilbert space contains $S_o$ as a subspace. Van Hove conjectured

[14] As van Hove pointed out, this implies that any normalized superposition of stationary states of the free field will have infinite average values for the square of the total and interaction Hamiltonians (when $g \neq 0$): $\left\langle H^2\right\rangle_\phi = \left\|H\phi\right\|^2 = +\infty$ and $\left\langle H_I^2\right\rangle_\phi = \left\|H_I\phi\right\|^2 = +\infty$.

that $H$ could be defined and diagonalized in this larger space using vectors that spanned the $S_g$ subspaces, which depend on the value of the coupling constant $g$. For different values of $g$, including the free field case of $g = 0$, the subspaces $S_g$ and $S_{g'}$ are *orthogonal* ($g \neq g'$). Van Hove's result that the space of free states is orthogonal to the space of interacting states is often cited with reference to his 1952 paper, but he had already anticipated this result in his 1951 paper.

In his 1952 paper, van Hove examined the case of a neutral scalar field that was in scalar interaction with infinitely heavy fixed point sources. In this case, an exact solution can be obtained and compared with the perturbative solution. Van Hove stated that the origin of the divergences in this case was due to the fact that the space of stationary states for the free field is orthogonal to the state space of the stationary states of the field interacting with the sources. The implications of this are nicely summarized in A. J. Coleman's (1953) review of van Hove's 1952 paper. "[Van Hove's result] suggests that there is no mathematical justification for using the interaction representation and the occasional successes of renormalization methods are lucky flukes." Van Hove showed that while the exact solution and the method of renormalized perturbations give the same $S$-matrix, they disagree on the unitary matrix $U(-\infty, t)$ for finite $t$. Since the original Dyson framework for QFT relies on doing series expansions using such unitary matrices, van Hove's result showed that these (unrenormalized) matrices do not exist. The nonexistence of the $U(-\infty, t)$ matrix was one of the results that Haag claimed to show in the quotation above.

### 2.6.2 Friedrichs' Work

While van Hove investigated ultraviolet divergences ($\vec{\mathbf{k}} = \infty$), Friedrichs investigated infrared divergences ($\vec{\mathbf{k}} = 0$) of bosons interacting with a source distribution in his 1953 book, *Mathematical Aspects of Quantum Field Theory*. He defined creation and annihilation operators in terms of the field operator and the field's canonical conjugate momentum operator. He then showed that there are representations of the creation and annihilation operators which satisfy their CCRs but for which the number operator is not defined. He called a representation of the field and its canonical conjugate momentum *amyriotic* if the total number operator is well defined and he called it *myriotic* if the total number operator is not well defined. The most striking feature of myriotic fields is that they do not possess a vacuum state, i.e., the no-particle state. In the case of infrared divergences, Friedrichs showed that the representation of the field operators is myriotic. This accounted for the problems associated with defining the Hamiltonian. Friedrichs showed that if one used myriotic fields that the Hamiltonian could be defined. This is roughly similar to van Hove's suggestion that the Hamiltonian could be defined if one allowed states that contained an infinite number of particles since myriotic fields do not have states that contain a finite number of particles. Friedrichs also showed that in certain cases the unitary operator $U(t)$ does not have a limit as $t \rightarrow \infty$, which again showed that there were problems constructing a unitary operator $U(t, \infty)$ for finite $t$.

### 2.6.3 Wightman and Schweber's Paper

There was bidirectional influence between Haag on the one hand and Wightman and Schweber on the other. Haag was given a preprint of the Wightman and Schweber's paper by Wightman when he was working on his 1955 paper, and Wightman and Schweber had access to some unpublished CERN lectures of Haag given in 1953. The Wightman and Schweber paper also highlighted the difficulties of making the Hamiltonian a well-defined operator. They showed that if a certain condition is satisfied, then the Hamiltonian of a system is well-defined for just one value of the coupling constant. Based on the work of van Hove, they showed that if a certain condition is satisfied while the uncoupled or free fields has a vacuum state, the equations of motion may show that the coupled system may not have a vacuum state. This undermined perturbation theory which assumes that both the free field and the coupled system have a vacuum state. This also showed that representations which require the existence of a vacuum state may be inconsistent with the equations of motion.

## 2.7 Haag's Proof in his 1955 Paper

The work cited above was known to Haag when he wrote his 1955 paper. The result in that paper which can be classified as "Haag's theorem" is that the field operators corresponding to the asymptotic, or free fields, which occur at time $t = \pm\infty$, and the field operators corresponding to the "actual fields," which occur at some finite time, belong to unitarily inequivalent representations of the CCRs. His proof is a *reductio ad absurdum*. Suppose the free fields and the "actual fields" *were* unitarily equivalent and satisfy the CCRs. Haag then showed two things. (1) The vacuum states of the two representations would have to be the same vacuum state. (2) It follows from (1) that the free fields must satisfy a different set of canonical commutation relations. Thus, the "actual fields" and the free fields belong to unitarily inequivalent representations of the CCRs.

The connection with van Hove's result is the following. Haag showed that the representations of the "actual" and free fields as operators acting on Hilbert spaces cannot be unitarily equivalent. Haag also assumed that these representations are irreducible. In modern operator theory, it is a well known mathematical fact that two irreducible representations are unitarily inequivalent if and only if they are orthogonal. Thus, van Hove's result that the Hilbert spaces of the stationary state space of the free fields is orthogonal to the stationary state space of the interacting field ($g \neq 0$) is contained within Haag's theorem because, as Haag noted (1955b, 31), he does not use a particular form of the Hamiltonian in his proof. In this sense, Haag generalized van Hove's result and it would be more appropriate to call Haag's theorem a generalized van Hove theorem because the essential physical points come from van Hove's work.

### 2.8 THE HALL-WIGHTMAN-GREENBERG PROOF OF HAAG'S THEOREM 1957-1959

As was mentioned earlier, Hall and Wightman did not find Haag's proof conclusive. In their 1957 paper, they proved a generalized Haag's Theorem that came in two parts. (1) Given two neutral scalar fields that are related at a finite time by a unitary transformation, that satisfy the CCRs, and that have unique normalizable vacuum states that are invariant under Euclidean transformations (translations and rotations); it then follows that the unitary transformation takes the vacuum state of the first field theory to the vacuum state of the second field theory multiplied by a constant whose absolute value is equal to one. (2) If two field theories satisfy the assumptions of (1) and they and their vacuum states are invariant under inhomogeneous Lorentz transformations and have no negative energy states, then the first four vacuum expectation values are equal for all times. This showed that no interaction could be modeled by the usual representation of the creation and annihilation operators where the first four vacuum expectation values differed from the free field values. As Hall and Wightman pointed out, their result is only valid up to the first four vacuum expectation values. Thus, even the Hall and Wightman result did not completely prove Haag's theorem. This was accomplished by Greenberg (1959), who used mathematical induction to prove that the first *n* vacuum expectation values are equal (for any positive integer *n*).

## 2.9 CONCLUSIONS

Determining who proved "Haag's theorem" depends on one's particular understanding of what the theorem is about. Since the Hamiltonian is the generator of time translations, van Hove's work, which showed that the interaction Hamiltonian and the total Hamiltonian are not well-defined on the space of stationary states of the free field, indicated that there would be problems constructing a unitary operator that connects the free fields to the interacting fields. Haag's proof can be seen as a generalization of van Hove's result, which shows only in a particular case that the stationary state space of the free field is orthogonal to the stationary state space of the field interacting with sources. Haag provided the first steps towards the more modern way of discussing the problem in terms of UIRs of the CCRs. In this way Haag's paper united the work of van Hove with Friedrichs' work on representations of the CCRs. If Haag's theorem is roughly understood to be a result that shows that an interacting field theory will have the same expectation values as a free field theory (one would expect them to be different), then the Hall-Wightman-Greenberg papers are a proof of that idea. However, if Haag's theorem is taken to be about how the interacting fields (or the interaction Hamiltonian) cannot be defined using the same canonical commutation relations (or Hilbert space, respectively) as the that of free fields, then the work of van Hove, Friedrichs, Wightman, Schweber, and Haag proved this in many cases. Though Haag's original 1955 proof and the "generalized" version of Haag's theorem in Hall and Wightman (1957) have some similarities, they have significant differences. Haag wanted to show that the free field representation and the "actual" or interacting field representation are not unitarily equivalent representations of the CCRs. Hall and Wightman showed that under certain conditions two field theories will, up to four vacuum expectation values, have the same values. The 1959 generalized version of the Hall-Wightman proof by Greenberg is used to show when a field theory will be equivalent to the free field theory without concerning itself with how the free fields might be constructed through some asymptotic condition. The next chapter will show explicitly how van Hove's results can be used to prove a van Hove-Haag theorem.

# Chapter Three

### 3.1 Introduction

The crisis of UIRs for the foundation of QFT has received remarkably little reaction from philosophers of physics. Apart from a few papers (Arageorgis, Earman, and Ruetsche 2002a) (Clifton and Halvorson 2001) (Kronz and Lupher 2005), there have been no published critical responses.[15] This is especially puzzling given the dramatic language used to characterize different UIRs as "*incommensurable*" (Arageorgis, Earman, and Ruetsche 2002b) and "*complementary*" (Clifton and Halvorson 2001) physical theories. The neglect of UIRs is partially due to the selective cases analyzed in some of these papers: the Unruh effect (Clifton and Halvorson 2001) (Arageorgis, Earman, and Ruetsche 2002a) and the thermodynamic limit of quantum statistical mechanics (Ruetsche 2003). This has left an impression among some people that UIRs arise only in physically questionable circumstances. In the case of the Unruh effect, the moral taken by some is that UIRs only occur when an inertial and accelerating observer's QFTs are compared. In the case of the thermodynamic limit, an "unphysical" limit is taken where the volume and the number of particles grows infinitely large, though their density is finite. Further, due to the complexities involved in understanding the algebraic framework, the appearance and properties of UIRs has not been transparent to philosophers of physics who only

[15] One critical response to UIRs for the foundations of QFT is (Wallace 2006).

have a background in canonical QFT. Indeed, while the algebraic framework is very useful for philosophers concerned about the conceptual analysis of QFT, most physicists ignore or are completely unaware of algebraic QFT. These issues have left the impression that UIRs can be ignored because they only occur in abstract mathematical settings for unusual physical situations.

This paper shows how UIRs occur in *canonical* QFT using a simple model. It shows that the two most puzzling aspects of the Unruh effect are in fact general features of UIRs in QFT that do not require inertial and accelerating observers for their appearance. This model shows how to construct a broad spectrum of UIRs in canonical QFT. In particular, I show that the key mathematical condition that leads to UIRs is the result of a physical condition: the invariance of the vacuum under spatial translations. It is this requirement that is the key to deriving a version of Haag's theorem for this model.

Awareness of the existence of a continuum of UIRs in QFT slowly entered the physics literature in the 1950s. The key feature that gives rise to them is that quantum fields have a countably infinite number of degrees of freedom, by contrast with quantum mechanics, which solely concerns systems having a finite number of degrees of freedom. Modulo some technical details, it is this difference that is crucial to proving the Stone-von Neumann theorem in quantum mechanics. That theorem guarantees that any representation of the canonical commutation relations (CCRs) is unitarily equivalent to the Schrödinger representation. One important application of this theorem is the unitary equivalence of wave mechanics and matrix mechanics. By contrast, quantum field theory concerns systems that have an infinite number of degrees of freedom, so the Stone-von Neumann theorem does not hold.

In the case of the Unruh effect (Unruh 1976), there are two key features: (1) when the field in Minkowski spacetime is in the vacuum state, inertial-frame observers do not detect particles, whereas a uniformly accelerating observer in the same region of spacetime detects a thermal flux of particles (sometimes described as an infinite number of particles), and (2) no total number operator is defined for the accelerating observer. These features have been used by some to argue that different observers have different particle concepts (Clifton and Halvorson 2001) or that “particles do not exist” (Davies 1984); they have their origin in the existence of UIRs of the CCRs.[16]

The analogues to (1) and (2) above were noted more than twenty years earlier by Friedrichs (1953), who made a distinction between particle and non-particle representations—the latter are also called “myriotic,” “strange,” and “non-Fock” representations. They have the distinguishing feature that the total number of particles is infinite for any state and no total number operator is definable.

> The impossibility of defining a number operator *N* for myriotic fields implies that *myriotic fields do not possess particle representations*, and hence, in particular, that *myriotic fields do not possess vacuum states*; this is perhaps the most striking property of such fields.” (Friedrichs 1953, 141)

[16] Details of the proof can be found in (Clifton and Halvorson 2001).

Friedrichs concluded that these types of representations cannot be particle representations because (1) no total number operator can be defined and (2) no vacuum state exists.[17] These are the main features of representations that are unitarily inequivalent to the Fock representation, and they play a crucial role below.

Before discussing (1) and (2), I present more of the history of the development of unitarily inequivalent representations in section 3.2, and then I provide an intuitive way to look at such representations in section 3.3 using the number occupation distribution where it is shown how the properties of myriotic fields are exhibited. In section 3.4, the crux of the paper, an important model is introduced and then used to illustrate some of the key properties of myriotic representations that are mentioned above. I also show how unitary equivalence can be established between two different representations and how that equivalence is broken if a certain condition is satisfied, in which case the total number operator becomes undefined and the probability of finding any state with a finite number of particles from the Fock representation is zero. Intuitively, this is usually taken to show that the new representation will have an infinite number of particles in every state. Finally, I show how a new vacuum is defined and how it is different from the *bare* or no-particle vacuum. Section 3.5 extends the model presented in section 3.4 to capture many different types of unitary inequivalence, such as van Hove's result that the state spaces associated with different values

[17] It turns out that (2) requires a caveat; there are different types of vacuums that can be constructed and a distinction must be made between the *bare* vacuum and the *dressed* vacuum.

of the coupling constant are “orthogonal.” I also prove a version of Haag’s theorem, which is a serious problem that arises in modeling free and interacting quantum fields in the same representation.[18] My version shows that the non-interacting representation (zero coupling constant) is unitarily *inequivalent* to an interacting representation (non-zero coupling constant). One unique feature of this proof is that time is not involved. In section 3.6, the Hamiltonian for this model will be examined. I show that the interaction Hamiltonian is not defined on the space where the free Hamiltonian is defined, which is a variant of Haag’s theorem. In section 3.7, it is shown how to create UIRs for different times, and masses, and in section 3.8 how the issue of unitary inequivalence percolates up to affect quantum fields. Possible critical comments are considered in section 3.9. In the concluding section, section 3.10, I provide a concise summary of the results that are obtained in sections 3.4-3.9.

### 3.2 A Brief History of Unitarily Inequivalent Representations

The first indication of UIRs, according to Wightman (1965b, 255), can be found in a lecture by von Neumann (2001) on the canonical anticommutation relations (CARs) given at Princeton in the winter of 1935/36. Von Neumann did not explicitly state that he had constructed UIRs in the lecture nor in another (1938) of his works that is often cited as containing examples of UIRs. According

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See section 3.4 below for more on this issue.

[18] As the previous chapter discussed, Haag’s theorem means different things to different people.

to Segal (1958, 35), von Neumann was aware of the existence of UIRs of the CARs, though he did not present the examples in (2001) and (1938) as such. Another example of UIRs can be extracted from the work of Bloch and Nordsieck (1937) who investigated the infrared catastrophe in the case of the coupling of the electron to a radiation field. They introduced a transformation similar to the one given in equation (3.4.2) below and showed that the probability of the electron emitting only a finite number of photons is zero. However, they did not consider whether the transformation could be unitarily implemented.

The earliest published example where a unitary transformation fails to exist in QFT is contained in the work Friedrichs (1953) mentioned earlier. Friedrichs was aware of the Bloch and Nordsieck paper. He investigated (1953, 99) the infrared catastrophe and extended their work by showing that not only does the probability that a field contains a finite number of particles converge to zero, but the expected number of particles becomes infinite. Around the same time, van Hove showed that the Hilbert spaces constructed for different values of the coupling constant were "orthogonal".[19] This result will be proved in section 5. However, van Hove did not consider whether the "orthogonal" Hilbert spaces were unitarily equivalent representations of the CCRs or CARs.

Wightman and Schweber (1955b) brought the work of Friedrichs and van Hove together using the mathematical techniques found in Friedrichs as well as

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[19] Orthogonality and unitary inequivalence are identical concepts if the representations are irreducible. A representation is irreducible if the only invariant subspaces under the action of all the operators in the algebra are the trivial subspace and the whole Hilbert space.

in Wightman and Gårding (1954a) (1954b). It was the first paper to give explicit examples of UIRs that involved the algebraic approach to quantum theory developed by von Neumann. They were the first authors to argue for the pervasiveness of UIRs in QFT. After discussing a representation of the CCRs that essentially has the form (3.4.2), they wrote (1955b, 824) that this example is "not [a] pathological phenomena whose construction requires mathematical trickery," but rather occurs "in the most elementary examples of field theory." The work of van Hove, Friedrichs, and Wightman and Schweber influenced Haag's (1955a) paper which contained "Haag's theorem." Haag also argued that UIRs must be examined in QFT and that they are the reason why there is no unitary operator that connects the free quantum field with the "actual" interacting quantum field. The simple model I develop in this paper will encompass the essential results of van Hove, Friedrichs, Wightman and Schweber, and Haag.

### 3.3 NUMBER OCCUPATION DISTRIBUTION

In this section, I will give the easiest and most intuitive starting point for investigating the nature of UIRs: the number occupation distribution. However, not all UIRs can be understood in terms of a number occupation distribution.[20] This idea originated with Friedrichs (1953), but Wightman's work with Gårding (1954a) (1954b) and Schweber (1955b) in the mid-1950's brought this idea to the attention of many physicists. The number occupation distribution is a way to classify all of the representations[21] of the CCRs (equation (3.3.1)) or the CARs (equation (3.3.2)) of the creation and annihilation operators.

$$\begin{aligned} \left[a_j, a_k^\dagger\right] &= a_j a_k^\dagger - a_k^\dagger a_j = \delta_{jk} \\ \left[a_j, a_k\right] &= \left[a_j^\dagger, a_k^\dagger\right] = 0 \end{aligned} \tag{3.3.1}$$

$$\begin{aligned} \left\{a_j, a_k^\dagger\right\} &= a_j a_k^\dagger + a_k^\dagger a_j = \delta_{jk} \\ \left\{a_j, a_k\right\} &= \left\{a_j^\dagger, a_k^\dagger\right\} = 0 \end{aligned} \tag{3.3.2}$$

$a_k^\dagger$ and $a_k$ are the creation and annihilation operators respectively and satisfy the CCRs or CARs. The indices $j, k = 1, 2, 3, \ldots$ represent degrees of freedom. In QFT, they are allowed to take values from the natural numbers, so there are a countably infinite number of degrees of freedom.

The number operator is usually defined as $N_k = a_k^\dagger a_k$.[22] When the number operator is applied to a state its value is the number of particles in state *k.* For each *k*, there is a number operator and they form a commuting set of projections. The representations of the CCRs or CARs can then be diagonalized with respect to the number operators. A simultaneous eigenstate $|n_1, n_2, \ldots\rangle$ of all the $N_k$ can be

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[20] For example, there are continuous representations; see footnote 26.

[21] Roughly, a representation of the CCRs is a realization of the creation and annihilation operators as linear unbounded operators defined on a common dense domain of a Hilbert space. See Emch (1972, 8-13) for more technical details. For those who are uneasy about dealing with unbounded operators, the creation and annihilation operators can be replaced by the Weyl operators and the CCRs by the Weyl form of the CCRs (Emch 1972, 92-93). These mathematical complications are unnecessary for the purposes of this paper.

characterized as a sequence of non-negative integers $\{n_1, n_2, ...\}$ where $n_1$ is the number of particles for the $k = 1$ number operator, $n_2$ is the number of particles for the $k = 2$ number operator, etc… Let $\eta = \{n_1, n_2, ...\}$ be an arbitrary sequence of non-negative integers and $\Gamma = \{\{n_1, n_2, ...\}, \{n_1', n_2', ...\}, ...\}$ be the set of all possible $\eta$-sequences. For each $\eta$-sequence there is an eigenstate $|n_1, n_2, ...\rangle$ and vice versa. If the $n_i$ in some $\eta$-sequence are allowed to be any non-negative integer value, then the particles it represents are bosons which satisfy the CCRs. When $n_i$ is restricted to the values 0 or 1, then the particles are fermions which satisfy the CARs. The set of possible eigenstates in either case is non-denumerable. For bosons there are $\aleph_0$ eigenvalues for each number operator $N_k$, which correspond to the number of particles, and, since each sequence is $\aleph_0$ long, there are $\aleph_0^{\aleph_0}$, or a continuum, of possible eigenstates.[23] For fermions there are 2 possible eigenvalues for each number operator $N_k$ and, since each sequence is $\aleph_0$ long, there are $2^{\aleph_0}$, or a continuum, of possible eigenstates. Since the set $\Gamma$ is non-denumerable, a separable Hilbert space cannot be constructed using

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[22] There are alternative ways to define the number operator that are more mathematically rigorous, but they will not be needed for our purposes.

[23] $\aleph_0$ is the cardinality of the positive integers; it is countably infinite. The continuum is the cardinality of the positive real numbers; it is non-countable or non-denumerable.

$\{|n_1,n_2,...\rangle,|n_1',n_2',...\rangle,...\}$ as a (orthogonal) basis.[24] A Hilbert space constructed from $\Gamma$'s basis vectors would be non-separable.

However, most physicists do not like working with non-separable Hilbert spaces (Streater and Wightman 2000, 85-86). Physicists typically select a countable subset as a basis for their Hilbert space such that each state contains only a finite total number of particles: $\left\{|n_1,n_2,...\rangle : \sum_{k=1}^{\infty} n_k < \infty\right\}$. Let $\Gamma_0$ be the subset of $\Gamma$ which contains all $\eta$-sequences of this type. The Fock representation's Hilbert space $\mathcal{H}(\Gamma_0)$ is obtained by using $\left\{|n_1,n_2,...\rangle : \sum_{k=1}^{\infty} n_k < \infty\right\}$ as basis vectors. This set of vectors contains the vacuum state which has no particles: $|0,0,0,...\rangle$.[25] The creation and annihilation operators act on these states as follows:

$$\begin{aligned} a_k^\dagger |n_1,n_2,...,n_{k-1},n_k,n_{k+1},...\rangle &= \sqrt{n_k+1}\,|n_1,n_2,...,n_{k-1},n_k+1,n_{k+1},...\rangle \\ a_k |n_1,n_2,...,n_{k-1},n_k,n_{k+1},...\rangle &= \sqrt{n_k}\,|n_1,n_2,...,n_{k-1},n_k-1,n_{k+1},...\rangle \end{aligned} \tag{3.3.3}$$

The $a_k^\dagger$ creation operator adds one particle to the *k*th position while the $a_k$ destruction operator removes one particle from the *k*th position. Notice that when any combination of creation and destruction operators is applied a finite number of times to vectors in the set $\left\{|n_1,n_2,...\rangle : \sum_{k=1}^{\infty} n_k < \infty\right\}$ that the resulting

[24] A space $\mathcal{H}$ is *separable* if it contains a countable basis such that any vector in $\mathcal{H}$ can be approximated by a linear combination of basis vectors to any accuracy. A *non-separable* space has an uncountable or non-denumerable basis.

[25] This vector corresponds to the sequence $\{0,0,0,...\} \in \Gamma_0$.

vector is still in the set. For example, the vector $|1,0,0,0,...\rangle$ is in $\mathcal{H}(\Gamma_0)$ and $a_3^\dagger a_2^\dagger a_1 |1,0,0,0,...\rangle = |0,1,1,0,...\rangle$ is also in $\mathcal{H}(\Gamma_0)$. Thus, the application of a finite number of $a_k^\dagger$, $a_k$ does not lead out of $\mathcal{H}(\Gamma_0)$. It is only through the application of an infinite number of $a_k^\dagger$ to vectors in $\mathcal{H}(\Gamma_0)$ that a vector can be constructed that is *not* in $\mathcal{H}(\Gamma_0)$. Such a vector does not have a finite number of particles (i.e., $\sum_{k=1}^{\infty} n_k = \infty$), for example, $|1,0,1,0,1,0,1,...\rangle \notin \mathcal{H}(\Gamma_0)$ and $\{1,0,1,0,1,0,1,...\} \notin \Gamma_0$.

Let the sequences that are left in $\Gamma$ that are not in $\Gamma_0$ be denoted as $\Gamma \setminus \Gamma_0$. For each sequence in $\Gamma \setminus \Gamma_0$, there is a vector that could be used as a basis vector for a new Hilbert space. But which sequences in $\Gamma \setminus \Gamma_0$ can be used to form a basis and which sequences correspond to vectors in unitarily inequivalent Hilbert spaces? The answer comes from a theorem of Wightman and Schweber (1955b).[26] Vectors generated from sequences that differ in only a finite number of places can be used as a basis for a Hilbert space. To build a new Hilbert space using sequences from $\Gamma \setminus \Gamma_0$, first choose a particular sequence in $\Gamma \setminus \Gamma_0$ and call it $\zeta$. Now find all sequences in $\Gamma \setminus \Gamma_0$ that differ from $\zeta$ in a finite number of places. The set of sequences furnish a set of vectors that

[26] There are some technical difficulties that that are involved in the proof. Wightman and Schweber show that every representation of the CARs is a direct sum of a discrete and a continuous representation. The definition of a discrete and continuous representation will be passed over here, but the interested reader can find a full description in Wightman and Schweber's paper. The theorem referred to above is valid for discrete representations of the CCRs and CARs. The conditions for two continuous representations of the CCRs or CARs to be

forms a basis for a separable Hilbert space. By a theorem of Wightman and Schweber, a representation of the CCRs on $\mathcal{H}(\Gamma_0)$ is unitarily *inequivalent* to a representation of the CCRs on $\mathcal{H}(\zeta)$ because each sequence in $\Gamma_0$ differs from every sequence in $\zeta$ in an infinite number of places and vice versa. Thus, an example of a representation of the CCRs or CARs has been created that is unitarily *inequivalent* to the Fock representation depending on whether every $n_i$ in each sequence in $\Gamma_0$ and $\zeta$ can take all non-negative integer values or only the values 0 and 1. There are a non-denumerable number of sequences in $\Gamma \setminus \Gamma_0$ that differ in an infinite number of places.[27] An equivalence relation can be defined on $\Gamma \setminus \Gamma_0$ such that each equivalence class contains all sequences that differ only in a finite number of places. The set of these equivalence classes $[\Gamma \setminus \Gamma_0]$ is non-denumerable. The vectors corresponding to the sequences in each equivalence class can be used as the basis to construct a Hilbert space. Thus, by defining the creation and annihilation operators on these Hilbert spaces one can build a continuum of UIRs of the CCRs and CARs from $[\Gamma \setminus \Gamma_0]$ that are unitarily inequivalent to the Fock representation and to each other. The $[\Gamma \setminus \Gamma_0]$ representations are myriotic or non-Fock representations. They have an infinite

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unitarily equivalent are much more complicated and involve the equivalence of measures and dimension functions.

[27] Another way to explain the reason that there are an uncountable number of UIRs is that there are an uncountable number of ways of choosing a countable subset from an uncountable set.

number of particles in them and they do not have the vacuum state in them since $\{0,0,0,...\} \notin \Gamma \setminus \Gamma_0$.

### 3.4 TRANSLATED OPERATORS

#### 3.4.1 Fourier Representation and the CCRs

The CCRs[28] in their discrete form are given by equation (3.3.1), however this form of the CCRs is not as helpful for doing practical calculations in what follows. The spatial distribution of a single particle state is represented by a wave packet and this accounts for the reason that the indices *j* and *k* are discrete. However, wave packet functions are Fourier transformable, thus the CCRs can be reformulated in the Fourier representation. The CCRs can be reformulated as:

$$\begin{aligned} \left[a(\boldsymbol{k}), a^{\dagger}(\boldsymbol{j})\right] &= \delta(\boldsymbol{k}-\boldsymbol{j}) \\ \left[a(\boldsymbol{k}), a(\boldsymbol{j})\right] &= \left[a^{\dagger}(\boldsymbol{k}), a^{\dagger}(\boldsymbol{j})\right] = 0 \end{aligned} \tag{3.4.1}$$

where $\boldsymbol{k}$ and $\boldsymbol{j}$ are momentum eigenvectors.[29] $a^{\dagger}(\boldsymbol{k})$ when applied to the vacuum state $|0\rangle_a$ creates a state $|1_k\rangle_a$ that has 1-particle with momentum $\boldsymbol{k}$.

The Fock representation can be built by applying all polynomials of $a^{\dagger}(\boldsymbol{k})$ to the

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[28] I will be focusing on the CCRs for the rest of the paper. With appropriate modifications, the model could be adapted to the CARs.

[29] There are no momentum eigenvectors with a sharp eigenvalue of *k* in a Hilbert space. To make these operators mathematically rigorous it is necessary to "smear" them with a square-integrable function $f(\boldsymbol{k})$ such that $a(f)=\int d^3\boldsymbol{k}\, f(\boldsymbol{k})\, a(\boldsymbol{k})$. One might reason that since $\boldsymbol{k}$ is a continuously varying parameter that there must be an uncountably infinite number of degrees of freedom. However, the degrees of freedom are reduced from being uncountably infinite to countably infinite by "smearing" the operators. Thus, (3.4.1) has the same number of degrees of freedom as (3.3.1).

vacuum state $|0\rangle_a$ (also called the *bare vacuum* and the *no-particle state*) and completing the resulting space of states. The term 'Fock representation' will only be used to refer to the *a*-representation which can have at most a finite number of particles.

### 3.4.2 Translated Creation and Annihilation Operators

One of the simplest ways to create different representations of the CCRs is to shift the creation and annihilation operators by complex numbers.[30] It is called a boson field translation. These new creation and annihilation operators are defined by

$$\begin{aligned} b(\boldsymbol{k}) &= a(\boldsymbol{k}) + c(\boldsymbol{k}) \\ b^{\dagger}(\boldsymbol{k}) &= a^{\dagger}(\boldsymbol{k}) + c^{*}(\boldsymbol{k}) \end{aligned} \tag{3.4.2}$$

where $c(\boldsymbol{k})$ is a complex number and $c^{*}(\boldsymbol{k})$ is its complex conjugate. A quick calculation shows that $b(\boldsymbol{k})$, $b^{\dagger}(\boldsymbol{k})$ satisfy the CCRs (3.4.1). Does there exists a unitary operator $U_{ab}$ that will transform $a(\boldsymbol{k})$, $a^{\dagger}(\boldsymbol{k})$ into $b(\boldsymbol{k})$, $b^{\dagger}(\boldsymbol{k})$, i.e.

[30] The shift model has been known in the physics literature for quite some time (for example, (Wightman 1965b, 248) and was discussed briefly in the philosophy of physics literature (Huggett and Weingard 1996, 306). However, very few details are usually provided. This paper uses variations of this model to build every type of UIR the author has seen mentioned in the physics literature.

$b(\boldsymbol{k}) = U_{ab} a(\boldsymbol{k}) U_{ab}^{-1}$ and $b^{\dagger}(\boldsymbol{k}) = U_{ab} a^{\dagger}(\boldsymbol{k}) U_{ab}^{-1}$?[31] There is and the generator $A$ of the unitary transformation $U_{ab}$ is

$$A = \int d^3\boldsymbol{k}\ \left(c^*(\boldsymbol{k}) a(\boldsymbol{k}) - c(\boldsymbol{k}) a^{\dagger}(\boldsymbol{k})\right),$$

where $U_{ab} = e^A$. A proof of this can be found in the appendix (section 3.11). A new vacuum state can be defined for the *b*-representation $|0\rangle_b := U_{ab}|0\rangle_a$ [32] as well as a new number operator $N_{b(\boldsymbol{k})} := b^{\dagger}(\boldsymbol{k}) b(\boldsymbol{k})$.[33] The *bare vacuum* state in the Fock representation is annihilated by the $a(\boldsymbol{k})$ operator, i.e. $a(\boldsymbol{k})|0\rangle_a = 0$, and it has zero particles, i.e. $N_{a(\boldsymbol{k})}|0\rangle_a = a^{\dagger}(\boldsymbol{k}) a(\boldsymbol{k})|0\rangle_a = 0$.[34] The new *b*-vacuum state behaves as one would expect it to when the new *b*-annihilation operator is applied to it

$$b(\boldsymbol{k})|0\rangle_b = U_{ab}\, a(\boldsymbol{k}) U_{ab}^{-1} U_{ab}|0\rangle_a = U_{ab}\, a(\boldsymbol{k})\,|0\rangle_a = 0$$

and likewise when the new number operator is applied to it

$$N_{b(\boldsymbol{k})}|0\rangle_b = b^{\dagger}(\boldsymbol{k}) b(\boldsymbol{k})|0\rangle_b = U_{ab} a^{\dagger}(\boldsymbol{k}) U_{ab}^{-1} U_{ab} a(\boldsymbol{k}) U_{ab}^{-1} U_{ab}|0\rangle_a = U_{ab} a^{\dagger}(\boldsymbol{k}) a(\boldsymbol{k})|0\rangle_a = 0.$$ [35]

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31 I will only focus on the unitary operators that transform the annihilation operators in the rest of the paper. It is a simple matter to verify that the unitary operator that transforms the annihilation operator also transforms the creation operator.

32 Notice that this new vacuum state is not the no-particle state. It is a superposition of particle states since the unitary operator has a creation operator in its generator. See below for more detail.

33 Non-vacuum states in the *b*-representation can be created by application of the shifted creation operator in the usual way: $|1_k\rangle_b := U_{ab}|1_k\rangle_a = U_{ab}\, a^{\dagger}(\boldsymbol{k})|0\rangle_a$.

34 The total number operator for the Fock representation or *a*-representation is $N_a := \int d^3k\, a^{\dagger}(\boldsymbol{k}) a(\boldsymbol{k})$.

35 The number operator also acts on non-vacuum states in the way one would expect $N_{b(\boldsymbol{k})}|n_k\rangle_b = b^{\dagger}(\boldsymbol{k}) b(\boldsymbol{k})|n_k\rangle_b = U_{ab}\, a^{\dagger}(\boldsymbol{k}) a(\boldsymbol{k})|n_k\rangle_a = n\, U_{ab}|n_k\rangle_a = n|n_k\rangle_b$.

The *b*-vacuum state is called the *dressed* or *physical vacuum*. A Fock space can be created from $|0\rangle_b$ by applying all polynomials of $b^\dagger(\boldsymbol{k})$ to it and completing it. Thus, $b(\boldsymbol{k})$, $b^\dagger(\boldsymbol{k})$ are also candidates for being the operators to build a quantum field theory.

One question worth considering is whether this unitary operator is proper or improper. The distinction between improper and proper unitary transformations can be found in the work of Barton (1963) and Roman (1969). Improper unitary transformations, are transformations that leave the CCRs, or CARs, unchanged but do *not* transform vectors in the representation into other vectors in the *same* Hilbert space. A proper unitary transformation does map vectors in a Hilbert space to other vectors in the *same* space. A unitary transformation *U* is *improper* when it only has vanishing matrix elements in a given representation, i.e., $\langle\varphi|U|\psi\rangle = 0$ for all non-zero fixed states $|\varphi\rangle$ and $|\psi\rangle$ in a Hilbert space. Proper and improper unitary transformations preserve the CCRs (CARs), but the improper unitary transformations take vectors out of the original Hilbert space and put them in an "orthogonal" Hilbert space. For example, the new vector $|\phi\rangle := U|\psi\rangle$ is "orthogonal" to the vectors in the original Hilbert space if the representations are connected by an improper unitary transformation since $\langle\psi|\phi\rangle = \langle\psi|U|\psi\rangle = 0$. If the unitary transformation is improper, the representations are classified as unitarily *in*equivalent. What the improper / proper distinction shows is that more than one Hilbert space is necessary in dealing with UIRs. Indeed, if a direct sum or tensor product of two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ corresponding to different UIRs is constructed, then the improper unitary operator $U:\mathcal{H}_1 \to \mathcal{H}_2$ becomes a proper unitary operator with the mapping $U:\mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ when the direct sum of the Hilbert spaces is used.[36]

For the case of the shifted creation and annihilation operators, the question is whether the unitary transformation $U_{ab}$ is proper or improper. To determine this it is useful to rewrite $U_{ab} = e^{\int d^3\boldsymbol{k}(c^*(\boldsymbol{k})a(\boldsymbol{k})-c(\boldsymbol{k})a^\dagger(\boldsymbol{k}))}$ in the form

$$U_{ab} = e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2} e^{-\int d^3k\, c(\boldsymbol{k})a^\dagger(\boldsymbol{k})} e^{\int d^3k\, c^*(\boldsymbol{k})a(\boldsymbol{k})}.$$ [37]

To simplify the discussion consider ${}_a\langle 0|U_{ab}|0\rangle_a$.[38] $U_{ab}|0\rangle_a$ reduces to

$$U_{ab}|0\rangle_a = e^{-\frac{1}{2}\int d^3\mathbf{k}\,|c(\mathbf{k})|^2} e^{-\int d^3k\, c(\mathbf{k})a^\dagger(\mathbf{k})}|0\rangle_a.$$

The *b*-vacuum state is a superposition of particle states from the *a*-representation:

[36] Van Hove (1952), who constructed the first explicit examples of UIRs, thought that an infinite tensor product of all the Hilbert spaces associated with UIRs for every value of the coupling constant was necessary for QFT, though he never developed this idea.
[37] This can be proved using the Baker-Campbell-Hausdorff formula
$e^A e^B = e^{A+B+\frac{1}{2!}[A,B]+\frac{1}{3!}[A,[A,B]]+\ldots}$.
[38] To prove that $U_{ab}$ is improper, the inner product of two arbitrary *a*-states ${}_a\langle m_k|U_{ab}|n_k\rangle_a$ should be shown to be equal to zero. However, as will be shown, the unitary inequivalence of the representations depends on the $e^{-\frac{1}{2}\int d^3\mathbf{k}\,|c(\mathbf{k})|^2}$ factor in $U_{ab}$. When $U_{ab}|n_k\rangle_a$ is expanded, each term will have a $e^{-\frac{1}{2}\int d^3\mathbf{k}\,|c(\mathbf{k})|^2}$ factor. Thus, it suffices to show that ${}_a\langle 0|U_{ab}|0\rangle_a = 0$ in order to show that the *b*-representation is unitarily inequivalent to the *a*-representation.

$$
\begin{aligned}
|0\rangle_b = U_{ab}|0\rangle_a &= e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2} e^{-\int d^3\boldsymbol{k}\,c(\boldsymbol{k})a^\dagger(\boldsymbol{k})}|0\rangle_a \\
&= e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2}|0\rangle_a - e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2}\int d^3\boldsymbol{k}\,c(\boldsymbol{k})|1_{\boldsymbol{k}}\rangle_a + \ldots
\end{aligned}
\tag{3.4.3}
$$

The *b*-representation will be unitarily equivalent to the *a*-representation as long as $\int d^3\boldsymbol{k}|c(\boldsymbol{k})|^2 < \infty$. If $\int d^3\boldsymbol{k}|c(\boldsymbol{k})|^2 = \infty$, then $e^{-\frac{1}{2}\int d^3\mathbf{k}\,|c(\mathbf{k})|^2} = 0$. When the *b*-vacuum is expanded in terms of basis vectors from the *a*-representation and $\int d^3\boldsymbol{k}|c(\boldsymbol{k})|^2$ diverges, the expansion coefficients $e^{-\frac{1}{2}\int d^3\mathbf{k}\,|c(\mathbf{k})|^2}$ in (3.4.3) vanish. The *b*-vacuum and the other *b*-states are no longer superpositions of states in the *a*-representation; i.e., they are no longer states in the Fock representation's Hilbert space. Thus, the unitary transformation is improper; it has vanishing matrix elements in the *a*-representation.

Is the *b*-representation a myriotic representation in Friedrichs' sense? The total number operator $N_b := \int d^3\boldsymbol{k}\, b^\dagger(\boldsymbol{k})b(\boldsymbol{k})$ can be rewritten in terms of the *a*-operators as

$$
\begin{aligned}
N_b &= \int d^3\boldsymbol{k}\left(a^\dagger(\boldsymbol{k}) + c^*(\boldsymbol{k})\right)\left(a(\boldsymbol{k}) + c(\boldsymbol{k})\right) \\
&= N_a + \int d^3\boldsymbol{k}\,c(\boldsymbol{k})a^\dagger(\boldsymbol{k}) + \int d^3\boldsymbol{k}\,c^*(\boldsymbol{k})a(\boldsymbol{k}) + \int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2.
\end{aligned}
\tag{3.4.4}
$$

The *b*-representation is unitarily inequivalent to the *a*-representation if and only if $\int d^3\boldsymbol{k}|c(\boldsymbol{k})|^2$ diverges. The latter implies that the total number operator for the *b*-representation is undefined since $N_b \approx \int d^3\boldsymbol{k}|c(\boldsymbol{k})|^2 = \infty$.[39]

[39] But this should not be viewed as a defect of the *b*-operators. The total number operator for the *a*-operators can also have this myriotic behavior; see the end of this section.

What is the probability of only finding a given finite number of *a*-particles in the *b*-vacuum, or any *b*-state for that matter? An *a*-state has *m* particles with momentum $\boldsymbol{k}$ is defined as $|m_k\rangle_a = \frac{\left(a^\dagger(\boldsymbol{k})\right)^m |0\rangle_a}{\sqrt{m!}}$. The probability of finding *m* *a*-particles with momentum $\boldsymbol{k}$ in the *b*-vacuum state is ${}_a\langle m_k|0\rangle_b = {}_a\langle m_k|U_{ab}|0\rangle_a = 0$ since $e^{-\frac{1}{2}\int d^3\mathbf{k}\,|c(\mathbf{k})|^2} = 0$. This is usually taken to mean that the *b*-vacuum has an infinite number of *a*-particles in it. The probability of finding only a finite number of *a*-particles in $|n_k\rangle_b$ is also zero: ${}_a\langle m_k|n_k\rangle_b = {}_a\langle m_k|U_{ab}|n_k\rangle_a = 0$.

A no-particle state does not exist in the *b*-representation's Hilbert space. The field translation induces boson condensation. In other words, one can think of each *b*-state as a state where *a*-particles are condensed. Thus, a myriotic representation has been created since no total number operator is defined for the *b*-representation and it does not have a bare vacuum state, though it does have a state that behaves like a vacuum state – the *dressed vacuum*. Thus, the particle content in a quantum field theory depends on the operators that satisfy the CCRs.

By changing the value of the complex number $c(\boldsymbol{k})$ in (3.4.2) we could create a different set of operators that would be unitarily inequivalent to the *a*-

representation *and* the *b*-representation if $\int d^3\boldsymbol{k}\left|c(\boldsymbol{k})\right|^2$ diverges.[40] In fact, a unitarily inequivalent representation could be created for each value of $c(\boldsymbol{k})$. Each of these representations would have a *different dressed vacuum*. Since $c(\boldsymbol{k})$ is a complex number, we could create a continuum of UIRs as well as a continuum of different vacuum states! This is vacuum polarization taken to the extreme.

### 3.4.3 Finite vs. Infinite Number of Degrees of Freedom

As noted above, an infinite number of degrees of freedom is crucial to generating UIRs. The Stone-von Neumann theorem guarantees that all irreducible representations of the Weyl form of the CCRs for systems with a *finite* number of degrees of freedom are unitarily equivalent to the Schrödinger representation. To see how the degrees of freedom allow for UIRs, we use the (3.3.1) form of the CCRs. The *b*-operators would then be defined in the following way.

$$\begin{aligned} b_k &= a_k + c_k \\ b_k^\dagger &= a_k^\dagger + c_k^* \end{aligned} \qquad (3.4.5)$$

[40] The construction of the unitary operator that would transform the representations for different values of $c(\boldsymbol{k})$ is very similar to constructions that are used in section 3.5 below that are physically more interesting. The condition for the unitary operator to be improper is also similar.

The unitary operator that transforms the *a*-operators into the *b*-operators is $U_{ab} = e^{\sum_k (c_k^* a_k - c_k a_k^\dagger)}$. When this is rewritten as $U_{ab} = e^{-\frac{1}{2}\sum_k |c_k|^2} e^{-\sum_k c_k a_k^\dagger} e^{\sum_k c_k^* a_k}$, the crucial first term has $\sum_k |c_k|^2$ in its exponent. If there are only a finite number of degrees of freedom, then the sum is up to some finite number *n*: $\sum_{k=1}^{n} |c_k|^2$. Since a finite sum of real numbers is finite, $\sum_{k=1}^{n} |c_k|^2 < \infty$. Thus, the representations are unitarily equivalent. However, if there is a countably infinite number of degrees of freedom, $\sum_{k=1}^{\infty} |c_k|^2$ can diverge, depending on the series of complex numbers.[41]

If $\int d^3\boldsymbol{k} \left|c(\boldsymbol{k})\right|^2$ diverges, then $e^{-\frac{1}{2}\sum_{k=1}^{\infty} |c_k|^2} = 0$ and the *b*-representation is unitarily inequivalent to the *a*-representation. Thus, the finite number of degrees of freedom is essentially what generates the uniqueness of a representation of the CCRs. It is why all quantum mechanical theories are unitarily equivalent to the Schrödinger representation, provided that the operators can be put in the Weyl form. A QFT has a countably infinite number of degrees of freedom since the fields are defined at each spacetime point and this allows for the radical freedom in constructing different quantum field theories.[42] In fact, as will be shown in the following the sections, there are a continuum of different ways to build a QFT.

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[41] If the sequence of complex numbers decreases rapidly enough, then the representations are unitarily equivalent.

[42] One might think that since spacetime has an uncountable number of points that a QFT would have an uncountably infinite number of degrees of freedom. However, fields must be "smeared"

### 3.4.4 Which Operators to Choose?

It might appear that the problem of defining a total number operator and the lack of a particle interpretation for the *b*-operators is merely a mathematical pathology. In that case, only the *a*-operators would ever be used to build a QFT. This position will be called *Fock-Representation chauvinism*. However, this argument is flawed for two reasons. (1) The problematic behavior of myriotic representations is not something endemic to those operators. Rather, it is due to applying operators from one representation to the states in a unitarily inequivalent representation. Consider the expectation value of the total number operator for the *a*-representation in the *b*-vacuum state

$$ {}_b\langle 0|N_a|0\rangle_b = \int d^3\boldsymbol{k}\; {}_a\langle 0|U_{ab}^{-1}a^\dagger(\boldsymbol{k})a(\boldsymbol{k})U_{ab}|0\rangle_a \,. $$

This can be rewritten as

$$ {}_b\langle 0|N_a|0\rangle_b = \int d^3\boldsymbol{k}\; {}_a\langle 0|a^\dagger(\boldsymbol{k})a(\boldsymbol{k})|0\rangle_a - \int d^3\boldsymbol{k}\, c(\boldsymbol{k})\; {}_a\langle 0|a^\dagger(\boldsymbol{k})|0\rangle_a $$

$$ -\int d^3\boldsymbol{k}\, c^*(\boldsymbol{k})\; {}_a\langle 0|a^\dagger(\boldsymbol{k})|0\rangle_a + \int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2 . $$

The last term diverges when the representations are unitarily inequivalent. Thus, the total number operator for the *a*-representation is not defined on the *b*-states. The expression ${}_b\langle 0|N_a|0\rangle_b = \infty$ is usually taken intuitively to mean that there is an infinite number of *a*-particles in the *b*-vacuum state. But the expectation value

---

out in much the same way that as the operators in (3.4.1) and, as discussed in footnote 29, this has the effect of reducing the number of degrees of freedom from being uncountably infinite to of the total number operator for the *b*-representation in the *a*-vacuum state is also infinite, since ${}_a\langle 0|N_b|0\rangle_a = \int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2$. The total number operator for the *b*-representation is not defined for states in the *a*-representation. Thus, there is not something inherent in the *a*-operators that should make them preferable to the *b*-operators. For UIRs, the total number operator of one representation is only defined on its own states. (2) The method of defining the *b*-operators in terms of the *a*-operators can be reversed so that the *a*-operators have neither a total number operator nor a bare vacuum state.[43] Suppose that the *b*-operators satisfy

$$\begin{aligned}&\left[b(\boldsymbol{k}), b^\dagger(\boldsymbol{l})\right] = \delta(\boldsymbol{k}-\boldsymbol{l})\\ &\left[b(\boldsymbol{k}), b(\boldsymbol{l})\right] = \left[b^\dagger(\boldsymbol{k}), b^\dagger(\boldsymbol{l})\right] = 0\end{aligned} \tag{3.4.6}$$

and they have a vacuum state $|0\rangle_b$ that is annihilated by the destruction operator $b(\boldsymbol{k})$, i.e., $b(\boldsymbol{k})|0\rangle_b = 0$. Then (3.4.2) can be reversed to define the *a*-operators in terms of the *b*-operators.

$$\begin{aligned}a(\boldsymbol{k}) &= b(\boldsymbol{k}) - c(\boldsymbol{k})\\ a^\dagger(\boldsymbol{k}) &= b^\dagger(\boldsymbol{k}) - c^*(\boldsymbol{k})\end{aligned} \tag{3.4.7}$$

The unitary operator that transforms the *b*-operators into the *a*-operators is the same except for exchanging the *a*-operators for the *b*-operators and a minus sign:

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being countably infinite; see footnote 49.

[43] What follows was inspired by (Wightman and Schweber 1955b). In their example of UIRs, if one set of operators has a bare vacuum state, then the other set of operators will not have a bare vacuum state and vice versa.

$$U_{ba} = e^{\int d^3\boldsymbol{k}(c(\boldsymbol{k})b^\dagger(\boldsymbol{k}) - c^*(\boldsymbol{k})b(\boldsymbol{k}))} = e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2} e^{\int d^3k\, c(\boldsymbol{k})b^\dagger(\boldsymbol{k})} e^{-\int d^3k\, c^*(\boldsymbol{k})b(\boldsymbol{k})}.$$

The representations will still be unitarily inequivalent, if $\int d^3\boldsymbol{k}\left|c(\boldsymbol{k})\right|^2$ diverges. But notice that the *a*-vacuum state is a superposition of *b*-states.

$$\begin{aligned} |0\rangle_a &= U_{ba}|0\rangle_b = e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2} e^{-\int d^3\boldsymbol{k}\, c(\boldsymbol{k})b^\dagger(\boldsymbol{k})}|0\rangle_b \\ &= e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2}|0\rangle_b - e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2}\int d^3\boldsymbol{k}\, c(\boldsymbol{k})|1_{\boldsymbol{k}}\rangle_b + \dots . \end{aligned} \tag{3.4.8}$$

So, it is no longer the *bare vacuum*. Also, the total number operator for the *a*-representation $N_a = \int d^3\boldsymbol{k}\, a^\dagger(\boldsymbol{k})a(\boldsymbol{k})$ when it is rewritten in terms of *b*-operators becomes

$$\begin{aligned} N_a &= \int d^3\boldsymbol{k}\left(b^\dagger(\boldsymbol{k}) - c^*(\boldsymbol{k})\right)\left(b(\boldsymbol{k}) - c(\boldsymbol{k})\right) \\ &= N_b - \int d^3\boldsymbol{k}\, c(\boldsymbol{k})b^\dagger(\boldsymbol{k}) - \int d^3\boldsymbol{k}\, c^*(\boldsymbol{k})b(\boldsymbol{k}) + \int d^3\boldsymbol{k}\left|c(\boldsymbol{k})\right|^2. \end{aligned} \tag{3.4.9}$$

The last term diverges, so the total number operator for the *a*-representation is undefined. Thus, the *a*-representation is myriotic if one starts with the *b*-operators and defines the *a*-operators in terms of them.

### 3.5 VAN HOVE AND HAAG'S THEOREM

Early work on UIRs was done by van Hove and Haag. Van Hove (1952) examined the interaction of a neutral scalar field with recoilless nucleon point sources. He showed that the Hilbert space of stationary states for the free field is "orthogonal" to the Hilbert space of the stationary states of the field interacting with the point sources. He also discussed a more general case in which a Hilbert space can be constructed for each value of a scalar dimensionless coupling constant $g$.[44] If $g=0$, then there is no interaction and the Hilbert space constructed for it contains the states of the free field. For different values of the coupling constant, $g$, the Hilbert spaces are orthogonal or unitarily *inequivalent*. I will now construct a proof using $b(\boldsymbol{k})$, $b^\dagger(\boldsymbol{k})$.

For the case of a neutral scalar field interacting with recoilless point nucleon sources the constant $c(\boldsymbol{k})$ can be written as $g\,c(\boldsymbol{k})$.[45] Thus, the *b*-operators can be written as

$$\begin{aligned} b(\boldsymbol{k}) &= a(\boldsymbol{k}) + g\,c(\boldsymbol{k}) \\ b^\dagger(\boldsymbol{k}) &= a^\dagger(\boldsymbol{k}) + g\,c^*(\boldsymbol{k}) \end{aligned} \tag{3.5.1}$$

Another set of creation and annihilation operators can be defined for a different coupling constant $g'$,

$$\begin{aligned} b'(\boldsymbol{k}) &= a(\boldsymbol{k}) + g'\,c(\boldsymbol{k}) \\ b'^\dagger(\boldsymbol{k}) &= a^\dagger(\boldsymbol{k}) + g'\,c^*(\boldsymbol{k}) \end{aligned} \tag{3.5.2}$$

These two Fock spaces are related by the unitary transformation

$$U_{bb'} = e^{\int d^3\boldsymbol{k}\,(g'-g)(c^*(\boldsymbol{k})a(\boldsymbol{k}) - c(\boldsymbol{k})a^\dagger(\boldsymbol{k}))}$$

and their vacuum states are defined as

[44] In the next section, I will discuss another van Hove result: that the free Hamiltonian and the interaction Hamiltonian are not defined on the same Hilbert space.

$$|0\rangle_b := U_g\,|0\rangle_a = e^{\int d^3\boldsymbol{k}\, g\,(c^*(\boldsymbol{k})a(\boldsymbol{k}) - c(\boldsymbol{k})a^\dagger(\boldsymbol{k}))}\,|0\rangle_a \text{, and}$$

$$|0\rangle_{b'} := U_{g'}\,|0\rangle_a = e^{\int d^3\boldsymbol{k}\, g'\,(c^*(\boldsymbol{k})a(\boldsymbol{k}) - c(\boldsymbol{k})a^\dagger(\boldsymbol{k}))}\,|0\rangle_a .$$

If $U_{bb'}$ is an improper unitary operator, then ${}_{b'}\langle 0|U_{bb'}|0\rangle_b = 0$ in which case the *b*-representation and the $b'$-representation are "orthogonal", or unitarily inequivalent. A quick calculation shows that ${}_{b'}\langle 0|U_{bb'}|0\rangle_b = {}_a\langle 0|U_{bb'}|0\rangle_a$.

$U_{bb'}$ can be rewritten as

$$U_{bb'} = e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,(g'-g)^2\,|c(\boldsymbol{k})|^2}\, e^{-\int d^3\boldsymbol{k}\,(g'-g)\,c(\boldsymbol{k})a^\dagger(\boldsymbol{k})}\, e^{\int d^3\boldsymbol{k}\,(g'-g)\,c^*(\boldsymbol{k})a(\boldsymbol{k})} .$$

If $\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2$ diverges, then the expansion coefficients $e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2}$ vanish which implies that ${}_{b'}\langle 0|U_{bb'}|0\rangle_b = 0$. Thus, the states of the representations are "orthogonal" to each other for different coupling constants. This completes the proof of van Hove's result. In this case, the physical reason that $\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2$ diverges is that $c(\boldsymbol{k})$ is defined as an integral over the source distribution $\rho(\boldsymbol{k})$ (Roman 1969, 126-127 and 134) (Emch 1972, 22). In the limiting case of a point source, $\rho(\boldsymbol{k}) \to 1$ (though any finite number will do), the integral $\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2$ diverges.

---

[45] Under these conditions, according to Wightman and Schweber (1955b, 824), $c(\boldsymbol{k})$ is a function of the coupling constant $g : c(\boldsymbol{k}, g) = g\,c(\boldsymbol{k}, 0)$. For notational simplicity, $c(\boldsymbol{k}, 0)$ is written as $c(\boldsymbol{k})$ in what follows.

Since the coupling constant is a real number $g \geq 0$, there are a continuum of unitarily inequivalent Fock spaces that can be constructed! For each value of the coupling constant, there is a pair of operators $b(\boldsymbol{k})$, $b^{\dagger}(\boldsymbol{k})$. To put the matter another way, I have parameterized the possible Fock spaces by the coupling constant, which effectively serves as a label of the unitarily inequivalent Fock spaces. The choice of creation and annihilation operator and of a vacuum state is *radically non-unique*.

One possible reaction to the above is that the UIRs only occur if $\int d^3\boldsymbol{k} \left|c(\boldsymbol{k})\right|^2$ diverges, so why not make it a requirement that $\int d^3\boldsymbol{k} \left|c(\boldsymbol{k})\right|^2 < \infty$? After all, the reason this condition is violated in van Hove's case is that he assumed that the particles are point-particles; an assumption that is not physically realistic. However, there can be other physical conditions which require that $\int d^3\boldsymbol{k} \left|c(\boldsymbol{k})\right|^2$ diverges, such as the one that is used to derive Haag's theorem. Haag's theorem has come to mean a number of different things, but there are two main theses that are connected with it: (1) there is no (proper) unitary operator that connects the free representation of the CCRs with the interacting representation of the CCRs and (2) the interacting Hamiltonian is not defined on the Hilbert space on which the free Hamiltonian is defined. I will show (1) in this section, and (2) is shown in the next section. The key condition for deriving Haag's theorem comes from physical considerations; namely, that the vacuum should be invariant under spatial translations. This is physically realistic (unlike van Hove's point sources) since the vacuum should be the same state regardless of where one is in spacetime.

**van Hove-Haag Theorem:** If the vacuum state is invariant under spatial translations, then the free representation $a(\boldsymbol{k})$, $a^{\dagger}(\boldsymbol{k})$ of the CCRs and the interacting representation $b(\boldsymbol{k})$, $b^{\dagger}(\boldsymbol{k})$ in (3.5.1) of the CCRs are unitarily inequivalent.[46]

> Proof: In the model above, the free representation is the case where $g=0$ in (3.5.1), and the creation and annihilation operators are the usual Fock representation operators (in our notation they are the *a*-operators). For interacting representations, $g \neq 0$. To determine whether the free Fock representation (*a*-representation) and an interacting Fock space (*b*-representation) are unitarily inequivalent, consider the unitary operator $U_{ab}$, where
>
> $$U_{ab} = e^{-\frac{1}{2}\int d^3\boldsymbol{k}\, g^2 \left|c(\boldsymbol{k})\right|^2} e^{-\int d^3k\, g\, c(\boldsymbol{k}) a^{\dagger}(\boldsymbol{k})} e^{\int d^3k\, g\, c^{*}(\boldsymbol{k}) a(\boldsymbol{k})}.$$
>
> The *b*-operators correspond to those at (3.5.1). If $\int d^3\boldsymbol{k} \left|c\left(\boldsymbol{k}\right)\right|^2$ diverges, then the two representations are unitarily inequivalent. It is the invariance of the vacuum under spatial translations that forces $\int d^3\boldsymbol{k} \left|c\left(\boldsymbol{k}\right)\right|^2$ to diverge. The vacuum state for the *b*-representation is defined as
>
> $$\left|0\right\rangle_b = e^{-\frac{1}{2}\int d^3\boldsymbol{k}\, g^2 \left|c(\boldsymbol{k})\right|^2} e^{-\int d^3k\, g\, c(\boldsymbol{k}) a^{\dagger}(\boldsymbol{k})} \left|0\right\rangle_a .$$

[46] The idea for the proof below came from (Umezawa 1993, 43).

Since $c(\boldsymbol{k})$ is a function of momentum, it can be Fourier transformed to be a function of position $\tilde{c}(\boldsymbol{x})$. But then $|0\rangle_b$ would be a function of position since each component of the superposition would have a coefficient of $e^{-\frac{g^2}{2}\int d^3\boldsymbol{x}\,|\tilde{c}(\boldsymbol{x})|^2}$. That would violate the requirement that the vacuum state be independent of position. For $c(\boldsymbol{k})$ to be the Fourier amplitude of a constant independent of spatial position, it must be that $c(\boldsymbol{k}) = c\,\delta(\boldsymbol{k})$ where $c$ is a complex number. Thus,

$\int d^3\boldsymbol{k}\left|c(\boldsymbol{k})\right|^2 = \int d^3\boldsymbol{k}\left|c\right|^2 \delta^2(\boldsymbol{k}) = \infty$ and $e^{-\frac{1}{2}\int d^3\boldsymbol{k}\,g^2\,|c(\boldsymbol{k})|^2} = 0$. QED

This van Hove-Haag theorem's proof is quite different from the typical textbook proofs of Haag's theorem. Most of them use Hall and Wightman's generalization of Haag's theorem as their basis. It is also quite different from Haag's original proof. This proof has the advantage of showing that the Fock representation (*a*-representation) is unitarily inequivalent to an interacting Fock space (*b*-representation), and that the assumption that the vacuum be invariant under spatial translations is the crucial element needed to generate the inequivalence. It goes beyond the typical proofs of Haag's theorem because it shows that for each non-zero value of the coupling constant there is a representation that is unitarily inequivalent to every other representation with a different coupling constant, which is one of van Hove's results!

### 3.6 THE FREE AND INTERACTION HAMILTONIAN

Haag's theorem is also associated with whether the interacting Hamiltonian is defined on the same space as the free Hamiltonian. This question was answered negatively by van Hove (1951). If the free and interacting Hamiltonians are defined on Hilbert spaces associated with UIRs, no unitary operator implements the dynamics from a noninteracting system to an interacting system. The free Hamiltonian is defined as $H_a = \int d^3\boldsymbol{k}\, \omega(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) a(\boldsymbol{k})$, where $\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + m^2}$ where $m$ is mass and $\boldsymbol{k}$ is the momentum operator. In the $b$-representation, the free Hamiltonian is $H_b = \int d^3\boldsymbol{k}\, \omega(\boldsymbol{k}) b^\dagger(\boldsymbol{k}) b(\boldsymbol{k})$, which can be rewritten as

$$
\begin{aligned}
H_b &= \int d^3\boldsymbol{k}\, \omega(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + \int d^3\boldsymbol{k}\, g\omega(\boldsymbol{k}) c^*(\boldsymbol{k}) a(\boldsymbol{k}) \\
&\quad + \int d^3\boldsymbol{k}\, g\omega(\boldsymbol{k}) c(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) + \int d^3\boldsymbol{k}\, g^2\omega(\boldsymbol{k}) \left|c(\boldsymbol{k})\right|^2 \\
&= H_a + H_i
\end{aligned}
\tag{3.6.1}
$$

where $H_i = \int d^3\boldsymbol{k}\, g\omega(\boldsymbol{k}) c^*(\boldsymbol{k}) a(\boldsymbol{k}) + \int d^3\boldsymbol{k}\, g\omega(\boldsymbol{k}) c(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) + \int d^3\boldsymbol{k}\, g^2\omega(\boldsymbol{k}) \left|c(\boldsymbol{k})\right|^2$.

The unitary operator that transforms the $a$-representation into the $b$-representation also transforms the free Hamiltonian of the $a$-representation to the free Hamiltonian of the $b$-representation

$$
\begin{aligned}
U_{ab} H_a U_{ab}^{-1} &= \int d^3\boldsymbol{k}\, \omega(\boldsymbol{k}) U_{ab}\, a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) U_{ab}^{-1} \\
&= \int d^3\boldsymbol{k}\, \omega(\boldsymbol{k}) U_{ab}\, a^\dagger(\boldsymbol{k}) U_{ab}^{-1} U_{ab}\, a(\boldsymbol{k}) U_{ab}^{-1} \\
&= \int d^3\boldsymbol{k}\, \omega(\boldsymbol{k}) b^\dagger(\boldsymbol{k}) b(\boldsymbol{k}) \\
&= H_b.
\end{aligned}
$$

$U_{ab}$ is improper and $H_b = \infty = H_i$, if $\int d^3\boldsymbol{k} \left| c(\boldsymbol{k}) \right|^2$ diverges. Thus, the interacting Hamiltonian $H_b$ is no longer defined on the Fock representation's Hilbert space, unlike the free Hamiltonian $H_a$. $H_b$ maps all non-zero vectors in the Fock representation's Hilbert space to vectors with infinite length; i.e., $H_b \left| \psi \right\rangle = \infty$ for all $\left| \psi \right\rangle$ in the Fock representation's Hilbert space – a hallmark of UIRs. This should make sense. There is an infinite number of *a*-particles in every *b*-state, so the free energy of the *b*-representation for *a*-states should also be infinite.[47] Since the Hamiltonian is the generator of time translations, there is no unitary dynamics from the space of the bare particles to the physical particles (another way to state Haag's theorem). It also follows that there is no proper unitary transformation that connects two Hamiltonians $H_{b(g)}, H_{b'(g')}$ with different coupling constants $(g \neq g')$ when $\int d^3\boldsymbol{k} \left| c(\boldsymbol{k}) \right|^2$ diverges. In fact, the result that $H_b$ is not defined on the Fock space on which $H_a$ is defined can be generalized: for two Hamiltonians $H_{b(g)}, H_{b'(g')}$ with different coupling constants $H_{b(g)}$ *cannot* be defined on the Fock space $H_{b'(g')}$ is defined on and vice versa when $\int d^3\boldsymbol{k} \left| c(\boldsymbol{k}) \right|^2$ diverges!

[47] When the free Hamiltonian is applied to the bare vacuum state, the energy is zero $H_a \left| 0 \right\rangle_a = \int d^3\boldsymbol{k}\, \omega(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) \left| 0 \right\rangle_a = 0$. But when the *b*-free Hamiltonian is applied to the bare vacuum state $H_b \left| 0 \right\rangle_a = \int d^3\boldsymbol{k}\, g\, \omega(\boldsymbol{k}) \left| 1_k \right\rangle_a + \int d^3\boldsymbol{k}\, g^2 \omega(\boldsymbol{k}) \left| c(\boldsymbol{k}) \right|^2 \left| 0 \right\rangle_a$ the last term diverges. Thus, the vectors, including the bare vacuum, of the *a*-representation are not *proper* vectors of $H_b$. However, when the *b*-free Hamiltonian is applied to a *b*-state

## 3.7 TIME, MASS, AND ALL THAT

The Hamiltonian provides a way to construct different types of UIRs. Our discussion of dynamics has not yet investigated the role of time. This section will show that UIRs can be constructed for different times. To incorporate time into the creation and annihilation operators we use $H_a$ (the free Hamiltonian of the *a*-representation) and $H_b$ (the free Hamiltonian of the *b*-representation):

$$\begin{aligned}
a(\boldsymbol{k},t) &= e^{iH_a t}\, a(\boldsymbol{k}) e^{-iH_a t} = e^{-i\omega(\boldsymbol{k})t}\, a(\boldsymbol{k}) \\
a^\dagger(\boldsymbol{k},t) &= e^{iH_a t}\, a^\dagger(\boldsymbol{k}) e^{-iH_a t} = e^{i\omega(\boldsymbol{k})t}\, a^\dagger(\boldsymbol{k}) \\
b(\boldsymbol{k},t) &= e^{iH_b t}\, b(\boldsymbol{k}) e^{-iH_b t} = e^{-i\omega(\boldsymbol{k})t}\, b(\boldsymbol{k}) \\
b^\dagger(\boldsymbol{k},t) &= e^{iH_b t}\, b^\dagger(\boldsymbol{k}) e^{-iH_b t} = e^{i\omega(\boldsymbol{k})t}\, b^\dagger(\boldsymbol{k}).
\end{aligned} \tag{3.7.1}$$

Does there exist a unitary operator that connects $a(\boldsymbol{k},t)$ with $a(\boldsymbol{k},t')$ ? Yes, the unitary operator that satisfies $a(\boldsymbol{k},t') = U_{a(\boldsymbol{k},t)a(\boldsymbol{k},t')}\, a(\boldsymbol{k},t) U^{-1}_{a(\boldsymbol{k},t)a(\boldsymbol{k},t')}$ is

$$U_{a(\boldsymbol{k},t)a(\boldsymbol{k},t')} = e^{i(t'-t)\int d^3\boldsymbol{k}\, \omega(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) a(\boldsymbol{k})} = e^{iH_a(t'-t)}.$$

---

$H_b |m_k\rangle_b = \int d^3\boldsymbol{k}\, \omega(\boldsymbol{k}) U_{ab}\, a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) |m_k\rangle_a = \int d^3\boldsymbol{k}\, m\,\omega(\boldsymbol{k}) U_{ab} |m_k\rangle_a = \int d^3\boldsymbol{k}\, m\,\omega(\boldsymbol{k}) |m_k\rangle_b$. In particular, the *b*-vacuum is annihilated by $H_b$ as it should be, but it is not annihilated by $H_a$.

The operator $U_{a(\boldsymbol{k},t)a(\boldsymbol{k},t')}$ is a proper unitary operator, so there is no difficulty implementing the dynamics in the *a*-representation for different times. Is there a unitary operator that connects $b(\boldsymbol{k},t)$ with $b(\boldsymbol{k},t')$ ? Yes, the unitary operator that satisfies $b(\boldsymbol{k},t')=U_{b(\boldsymbol{k},t)b(\boldsymbol{k},t')}\,b(\boldsymbol{k},t)U^{-1}_{b(\boldsymbol{k},t)b(\boldsymbol{k},t')}$ is

$$U_{b(\boldsymbol{k},t)b(\boldsymbol{k},t')}=e^{i(t'-t)\int d^3\boldsymbol{k}\,\omega(\boldsymbol{k})\,b^\dagger(\boldsymbol{k})\,b(\boldsymbol{k})}=e^{iH_b(t'-t)}.$$

$U_{b(\boldsymbol{k},t)b(\boldsymbol{k},t')}$ can be rewritten as

$$U_{b(\boldsymbol{k},t)b(\boldsymbol{k},t')}=e^{i(t'-t)\int d^3\boldsymbol{k}\,\omega(\boldsymbol{k})\left|c(\boldsymbol{k})\right|^2}\,e^{i(t'-t)\int d^3\boldsymbol{k}\,\omega(\boldsymbol{k})\left[H_a+c^*(\boldsymbol{k})a(\boldsymbol{k})+c(\boldsymbol{k})a^\dagger(\boldsymbol{k})\right]}.$$

If $\int d^3\boldsymbol{k}\,\omega(\boldsymbol{k})\left|c(\boldsymbol{k})\right|^2$ diverges, then $e^{i(t'-t)\int d^3\boldsymbol{k}\,\omega(\boldsymbol{k})\left|c(\boldsymbol{k})\right|^2}$ becomes undefined in which case $U_{b(\boldsymbol{k},t)b(\boldsymbol{k},t')}$ becomes an improper unitary operator.[48] Thus, if $\int d^3\boldsymbol{k}\,\omega(\boldsymbol{k})\left|c(\boldsymbol{k})\right|^2$ diverges, then a *different* UIR must be used at every time! If the coupling constant $g$ is incorporated as in (3.5.1), then a UIR would have to be used at each instant for describing the *same* interaction. This case helps substantiate the claim that a different myriotic representation must be used at each different time for interactions (Wightman 1965b, 255). The Fock spaces have been parameterized by time and thus a continuum of UIRs has been generated.

[48] The reason this operator becomes undefined instead of going to zero as it did in the previous cases is that the exponent has an *i* in it. The exponential then has the form cos x + *i* sin x, and this function becomes undefined as x goes to infinity.

Since $\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k^2} + m^2}$ , UIRs can also be generated for different values of mass. I will show how to construct UIRs for different masses by formulating the creation and annihilation operators with different masses for both the *a*-representation and the *b*-representation where $\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k^2} + m^2}$ and $\omega'(\boldsymbol{k}) = \sqrt{\boldsymbol{k^2} + M^2}$ .

$$\begin{aligned} a(\boldsymbol{k},m,t) &= e^{-i\omega(\boldsymbol{k})t}\, a(\boldsymbol{k}),\ a^\dagger(\boldsymbol{k},m,t) = e^{i\omega(\boldsymbol{k})t}\, a^\dagger(\boldsymbol{k}) \\ a(\boldsymbol{k},M,t) &= e^{-i\omega'(\boldsymbol{k})t}\, a(\boldsymbol{k}),\ a^\dagger(\boldsymbol{k},M,t) = e^{i\omega'(\boldsymbol{k})t}\, a^\dagger(\boldsymbol{k}) \\ b(\boldsymbol{k},m,t) &= e^{-i\omega(\boldsymbol{k})t}\, b(\boldsymbol{k}),\ b^\dagger(\boldsymbol{k},m,t) = e^{i\omega(\boldsymbol{k})t}\, b^\dagger(\boldsymbol{k}) \\ b(\boldsymbol{k},M,t) &= e^{-i\omega'(\boldsymbol{k})t}\, b(\boldsymbol{k}),\ b^\dagger(\boldsymbol{k},M,t) = e^{i\omega'(\boldsymbol{k})t}\, b^\dagger(\boldsymbol{k}) \end{aligned} \tag{3.7.2}$$

The unitary operator that connects $a(\boldsymbol{k},m,t)$ with $a(\boldsymbol{k},M,t)$ is

$$U_{a(\boldsymbol{k},m,t)a(\boldsymbol{k},M,t)} = e^{it\int d^3\boldsymbol{k}\left(\omega'(\boldsymbol{k})-\omega(\boldsymbol{k})\right)a^\dagger(\boldsymbol{k})a(\boldsymbol{k})} = e^{i\left(H_{a(M)}-H_{a(m)}\right)t} .$$

This unitary operator is proper, so there is no problem representing different masses in the *a*-representation. The unitary operator that connects $b(\boldsymbol{k},m,t)$ with $b(\boldsymbol{k},M,t)$ is

$$U_{b(\boldsymbol{k},m,t)b(\boldsymbol{k},M,t)} = e^{it\int d^3\boldsymbol{k}\left(\omega'(\boldsymbol{k})-\omega(\boldsymbol{k})\right)b^\dagger(\boldsymbol{k})b(\boldsymbol{k})} = e^{i\left(H_{b(M)}-H_{b(m)}\right)t} .$$

It can be rewritten as

$$U_{b(\boldsymbol{k},m,t)b(\boldsymbol{k},M,t)} = e^{it\int d^3\boldsymbol{k}\left(\omega'(\boldsymbol{k})-\omega(\boldsymbol{k})\right)\left|c(\boldsymbol{k})\right|^2}\, e^{it\int d^3\boldsymbol{k}\left(\omega'(\boldsymbol{k})-\omega(\boldsymbol{k})\right)\left[H_a+c^*(\boldsymbol{k})a(\boldsymbol{k})+c(\boldsymbol{k})a^\dagger(\boldsymbol{k})\right]} .$$

If $\int d^3\boldsymbol{k}\left(\omega'(\boldsymbol{k})-\omega(\boldsymbol{k})\right)\left|c(\boldsymbol{k})\right|^2$ diverges, then $e^{i(t'-t)\int d^3\boldsymbol{k}\,\omega(\boldsymbol{k})\left|c(\boldsymbol{k})\right|^2}$ becomes undefined in which case $U_{b(\boldsymbol{k},t)b(\boldsymbol{k},t')}$ becomes an improper unitary operator. In this case, the continuum of UIRs has been parameterized by different masses.

### 3.8 QUANTUM FIELDS

The existence of UIRs also has consequences for the ontology of quantum fields. There is an equal time CCR (ETCCR) for a quantum field $\varphi(\boldsymbol{x},t)$ and its conjugate momentum field $\pi(\boldsymbol{x},t)$.[49]

$$
\begin{aligned}
&\left[\varphi(\boldsymbol{x},t),\pi(\boldsymbol{x}',t)\right]=i\delta(\boldsymbol{x}-\boldsymbol{x}') \\
&\left[\varphi(\boldsymbol{x},t),\varphi(\boldsymbol{x}',t)\right]=\left[\pi(\boldsymbol{x},t),\pi(\boldsymbol{x}',t)\right]=0
\end{aligned}
\tag{3.8.1}
$$

The reason the ETCCR looks remarkably similar to the CCRs that were discussed previously is that the quantum field and its conjugate momentum field are built from the creation and annihilation operators. The simplest example of a quantum field is a free neutral (bosonic) scalar field. It is defined by:

[49] Quantum fields defined at a spacetime point are not mathematically well-defined, so it is necessary to "smear" the fields spatially with a test function $f$, which for simplicity is assumed to be square integrable, such that $\varphi(f,t)=\int d\boldsymbol{x}\,\varphi(\boldsymbol{x},t)f(\boldsymbol{x})$ and $\pi(f,t)=\int d\boldsymbol{x}\,\pi(\boldsymbol{x},t)f(\boldsymbol{x})$.

$$\begin{aligned}
\varphi_a(\boldsymbol{x},t) &= \frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\,\frac{1}{\sqrt{2\omega(\boldsymbol{k})}}\left(a(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}+a^\dagger(\boldsymbol{k})e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\right)\\
\pi_a(\boldsymbol{x},t) &= \frac{\partial}{\partial t}\varphi^\dagger(\boldsymbol{x},t) \qquad\qquad (3.8.2)\\
&= \frac{-i}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\sqrt{\frac{\omega(\boldsymbol{k})}{2}}\left(a(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}-a^\dagger(\boldsymbol{k})e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\right).
\end{aligned}$$

It should therefore come as no surprise that UIRs can arise from the ETCCRs. One could construct quantum fields using the *b*-operators instead of the *a*-operators in (3.8.2).

$$\begin{aligned}
\varphi_b(\boldsymbol{x},t) &= \frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\,\frac{1}{\sqrt{2\omega(\boldsymbol{k})}}\left(b(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}+b^\dagger(\boldsymbol{k})e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\right)\\
\pi_b(\boldsymbol{x},t) &= \frac{\partial}{\partial t}\varphi^\dagger(\boldsymbol{x},t) \qquad\qquad (3.8.3)\\
&= \frac{-i}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\sqrt{\frac{\omega(\boldsymbol{k})}{2}}\left(b(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}-b^\dagger(\boldsymbol{k})e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\right)
\end{aligned}$$

The unitary operator $U_{ab}=e^{\int d^3\boldsymbol{k}(c^*(\boldsymbol{k})a(\boldsymbol{k})-c(\boldsymbol{k})a^\dagger(\boldsymbol{k}))}$ transforms $\varphi_a(\boldsymbol{x},t)$ and $\pi_a(\boldsymbol{x},t)$ into $\varphi_b(\boldsymbol{x},t)$ and $\pi_b(\boldsymbol{x},t)$ because

$$\begin{aligned}
\varphi_b(\boldsymbol{x},t) &= U_{ab}\varphi(\boldsymbol{x},t)U_{ab}^{-1}\\
&= \frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\,\frac{1}{\sqrt{2\omega(\boldsymbol{k})}}\left(U_{ab}\,a(\boldsymbol{k})U_{ab}^{-1}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}+U_{ab}a^\dagger(\boldsymbol{k})U_{ab}^{-1}e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\right)\\
&= \frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\,\frac{1}{\sqrt{2\omega(\boldsymbol{k})}}\left(b(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}+b^\dagger(\boldsymbol{k})e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\right).
\end{aligned}$$

If $\int d^3\boldsymbol{k}\left|c(\boldsymbol{k})\right|^2$ diverges, then $\varphi_a(\boldsymbol{x},t),\pi_a(\boldsymbol{x},t)$ and $\varphi_b(\boldsymbol{x},t),\pi_b(\boldsymbol{x},t)$ are UIRs of quantum fields.

The $\varphi_b(\boldsymbol{x},t)$ field differs from the free $\varphi_a(\boldsymbol{x},t)$ field; it is a superposition of fields. If one expands the $\varphi_b(\boldsymbol{x},t)$ field in terms of *a*-operators, then

$$\begin{aligned}\varphi_b(\boldsymbol{x},t) &= \frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\frac{1}{\sqrt{2\omega(\boldsymbol{k})}}\left(a(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}+a^\dagger(\boldsymbol{k})e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\right)\\ &+\frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\frac{1}{\sqrt{2\omega(\boldsymbol{k})}}\left(c(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}+c^*(\boldsymbol{k})e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\right)\\ &= \varphi_a(\boldsymbol{x},t)+\varphi_c(\boldsymbol{x},t)\end{aligned}$$

where

$$\varphi_c(\boldsymbol{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\frac{1}{\sqrt{2\omega(\boldsymbol{k})}}\left(c(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}+c^*(\boldsymbol{k})e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\right).$$

The $\varphi_c(\boldsymbol{x},t)$ field is a classical field since it has no operators in it. It should come as no surprise that if one introduces a coupling constant as was done in (3.5.1) that UIRs can be constructed for each value of *g*. The same operator that transforms the *b*-operators with different values of the coupling constant would also transform the quantum fields $\varphi_b(\boldsymbol{x},g,t)$, $\pi_b(\boldsymbol{x},g,t)$ into $\varphi_b(\boldsymbol{x},g',t),\pi_b(\boldsymbol{x},g',t)$. For every coupling constant, a UIR of a quantum field would be necessary as long as $\int d^3\boldsymbol{k}\left|c(\boldsymbol{k})\right|^2$ diverges.[50] It is easy to see that the van Hove-Haag theorem proved in section 3.5 can be modified to show that the

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50 In this case, $\varphi_b(\boldsymbol{x},g,t)=\varphi_a(\boldsymbol{x},t)+g\varphi_c(\boldsymbol{x},t)$.

free quantum field $\varphi_a(\boldsymbol{x},t)$ will be unitarily inequivalent to an interacting quantum field $\varphi_b(\boldsymbol{x},g,t)$.

**van Hove-Haag Theorem for Quantum Fields:** If the vacuum state is invariant under spatial translations, then the representation $\varphi_a(\boldsymbol{x},t)$, $\pi_a(\boldsymbol{x},t)$ of the ETCCRs for a free neutral scalar quantum field and the representation $\varphi_b(\boldsymbol{x},g,t), \pi_b(\boldsymbol{x},g,t)$ of the ETCCRs for an interacting quantum field are unitarily inequivalent.

The proof is essentially the same as in section 3.5.

The different types of unitary inequivalence shown in section 3.7 also hold at the level of quantum fields. The time dependence of the field can be incorporated into the *b*-operators using (3.7.1).

$$\begin{aligned}
\varphi_b(\boldsymbol{x},t) &= \frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\,\frac{1}{\sqrt{2\omega(\boldsymbol{k})}}\left(b(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + b^\dagger(\boldsymbol{k},t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right) \\
\pi_b(\boldsymbol{x},t) &= \frac{\partial}{\partial t}\varphi^\dagger(\boldsymbol{x},t) \\
&= \frac{-i}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\,\sqrt{\frac{\omega(\boldsymbol{k})}{2}}\left(b(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - b^\dagger(\boldsymbol{k},t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)
\end{aligned}$$

The operator that transforms $b(\boldsymbol{k},t)$ into $b(\boldsymbol{k},t')$ also transforms $\varphi_b(\boldsymbol{x},t), \pi_b(\boldsymbol{x},t)$ into $\varphi_b(\boldsymbol{x},t'), \pi_b(\boldsymbol{x},t')$. Thus, if $\int d^3\boldsymbol{k}\,\omega(\boldsymbol{k})\left|c(\boldsymbol{k})\right|^2$ diverges, a different UIR must be used at every instant of time for the $\varphi_b(\boldsymbol{x},t)$ quantum field!

With minor modifications[51], it is also possible to construct UIRs of the $\varphi_b(\boldsymbol{x},t)$ field for different masses. Thus, there is a continuum of unitarily inequivalent ways to build even a simple neutral scalar field.

All of the results at the creation / annihilation operator level of the CCRs have been shown to hold at the quantum field level of the ETCCRs. Is there anything new that occurs once quantum fields make an appearance? Yes, there is a new type of myriotic property that occurs which I call *hypermyriotic*. Now that fields appear one can consider a case similar to ${}_a\langle 0|N_b|0\rangle_a$, namely the two-point function using $\varphi_b(\boldsymbol{x},t)$ and the *a*-vacuum state $|0\rangle_a$: ${}_a\langle 0|\varphi_b(\boldsymbol{x},t)\varphi_b(\mathbf{y},t)|0\rangle_a$.

When this expression is rewritten using $\varphi_b(\boldsymbol{x},t)=\varphi_a(\boldsymbol{x},t)+\varphi_c(\boldsymbol{x},t)$ the two-point function becomes

$$\begin{aligned}{}_a\langle 0|\varphi_b(\boldsymbol{x},t)\varphi_b(\mathbf{y},t)|0\rangle_a &= {}_a\langle 0|\varphi_a(\boldsymbol{x},t)\varphi_a(\mathbf{y},t)|0\rangle_a + {}_a\langle 0|\varphi_b(\boldsymbol{x},t)\varphi_c(\mathbf{y},t)|0\rangle_a \\ &+ {}_a\langle 0|\varphi_c(\boldsymbol{x},t)\varphi_b(\mathbf{y},t)|0\rangle_a + {}_a\langle 0|\varphi_c(\boldsymbol{x},t)\varphi_c(\mathbf{y},t)|0\rangle_a\end{aligned}$$

The last term is the two-point function of a classical field with a quantum vacuum state. The classical two-point function can be written as:

$$\begin{aligned}{}_a\langle 0|\varphi_c(\boldsymbol{x},t)\varphi_c(\mathbf{y},t)|0\rangle_a &= \frac{1}{(2\pi)^3}\int d^3\boldsymbol{k}\,\frac{1}{2\omega(\boldsymbol{k})}\int d^3\boldsymbol{k}\;{}_a\langle 0|\{c(\boldsymbol{k})c(\boldsymbol{k})e^{i\boldsymbol{k}\cdot(\boldsymbol{x}+\mathbf{y})}+ \\ &|c(\boldsymbol{k})|^2\left(e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\mathbf{y})}+e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\mathbf{y})}\right)+c^*(\boldsymbol{k})c^*(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}+\mathbf{y})}\}|0\rangle_a .\end{aligned}$$

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51 For different masses, the unitary operator would be $U_{\varphi(\boldsymbol{x},m,t)\varphi(\boldsymbol{x},M,t)}=\left(\frac{\omega(\boldsymbol{k})}{\omega'(\boldsymbol{k})}\right)^{1/4}U_{b(\boldsymbol{x},m,t)b(\boldsymbol{x},M,t)}$.

Notice that the middle term has the following form.

$$ {}_a\langle 0|\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2\left(e^{i\boldsymbol{k}\cdot(\mathbf{x}-\mathbf{y})}+e^{-i\boldsymbol{k}\cdot(\mathbf{x}-\mathbf{y})}\right)|0\rangle_a $$

The *b*-representation is unitarily inequivalent to the *a*-representation when $\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2$ diverges and thus ${}_a\langle 0|\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2\left(e^{i\boldsymbol{k}\cdot(\mathbf{x}-\mathbf{y})}+e^{-i\boldsymbol{k}\cdot(\mathbf{x}-\mathbf{y})}\right)|0\rangle_a$ will diverge which entails that both ${}_a\langle 0|\varphi_c(\boldsymbol{x},t)\varphi_c(\mathbf{y},t)|0\rangle_a$ and ${}_a\langle 0|\varphi_b(\boldsymbol{x},t)\varphi_b(\mathbf{y},t)|0\rangle_a$ diverge. In other words, for UIRs not only does ${}_a\langle 0|N_b|0\rangle_a$ diverge but the two-point function ${}_a\langle 0|\varphi_b(\boldsymbol{x},t)\varphi_b(\mathbf{y},t)|0\rangle_a$ diverges as well. $\varphi_b(\boldsymbol{x},t)$ cannot be applied to vector states in the *a*-representation. In fact, the two-point function ${}_b\langle 0|\varphi_b(\boldsymbol{x},t)\varphi_b(\mathbf{y},t)|0\rangle_b$ of $\varphi_b(\boldsymbol{x},t)$ with its own vacuum state $|0\rangle_b$ also diverges since ${}_b\langle 0|\varphi_c(\boldsymbol{x},t)\varphi_c(\mathbf{y},t)|0\rangle_b = {}_a\langle 0|\varphi_c(\boldsymbol{x},t)\varphi_c(\mathbf{y},t)|0\rangle_a$ and the latter term diverges.

If the behavior of ${}_a\langle 0|N_b|0\rangle_a$ is supposed to license the inference that "particles do not exist" or that we need "complementary" particle concepts, then the behavior of ${}_a\langle 0|\varphi_b(\boldsymbol{x},t)\varphi_b(\mathbf{y},t)|0\rangle_a$ could also be used to argue that quantum fields do not exist or that we need complementary quantum fields concepts. The complementary nature of particles or fields seems dubious since $[N_a, N_b]=0$ and $[\varphi_a(\boldsymbol{x},t),\varphi_b(\boldsymbol{x},t)]=0$. The claim that particles or fields do not exist also is questionable since the reason that the expectation values ${}_a\langle 0|N_b|0\rangle_a$ and ${}_a\langle 0|\varphi_b(\boldsymbol{x},t)\varphi_b(\mathbf{y},t)|0\rangle_a$ diverge is because the operators and states of two UIRs are being combined into an expression that is not mathematically well-defined.

## 3.9 Criticisms

### 3.9.1 Representations: One versus the Many

One might look at the examples constructed so far and wonder if there is a set of criteria which would select one representation as being a better representation of physical reality than the others. It could be argued that one should only use the Fock representation—i.e., stick with the *a*-operators and be a *Fock-Representation* chauvinist. This is a topic that is worthy of further research, but I will limit myself to a few brief remarks. First, it was shown above that there was nothing inherently superior in the definition of the *a*-operators with respect to the *b*-operators because the *a*-operators could be defined in terms of the *b*-operators and then the *a*-operators would have the properties of a myriotic representation. Second, if one only uses the *a*-operators and assumes that the vacuum is invariant under spatial translations, then Haag's theorem shows that one cannot use the Fock representation to model interactions. Third, if one only works with the *free* scalar field, UIRs can still occur for different masses (see (Reed and Simon 1975, 233-235)). Thus, one cannot avoid UIRs by only working with the *a*-operators. If one wants to model van Hove-type interactions, then some of the freedom of choosing a representation will be eliminated by specifying a certain mass, and coupling constant. But even then UIRs will be necessary if one wants to know how the system evolves because a different UIR will be required at each instant of time. Not only are there many different ways to build quantum fields, but some of these fields will have states that can be given a particle interpretation while other myriotic fields do not. While many have used the Unruh effect to undercut the status of the particle concept in QFT in order to emphasize the primacy of the quantum field, unitary inequivalence shows that even the notion of the quantum field is not unique.

### 3.9.2 Toy Model Objection

It could be argued that no conclusions about QFT can be drawn from such a simple model. A more realistic model of interactions in QFT would offer a better guide to interpreting QFT. Unfortunately, we do not possess a "realistic" interacting model in QFT. Some progress has been made in constructive QFT for modeling interactions in spacetime dimensions less than four, but it is not clear that they would satisfy the criteria for being "realistic." In a recent article on QFT, Strocchi argues that physicists do not have such a model.

> [A]fter more than seven decades no non trivial (even non realistic) model in four (space time) dimensions is under non perturbative control. Actually, the prototypic model of self interacting scalar field, which is used in most textbooks for developing (non trivial) perturbation theory, has been proved to be trivial (namely the renormalized coupling constant vanishes when the ultraviolet cutoff is removed) under general conditions, when treated non perturbatively. This means that in general the perturbative expansion is not reliable and in general one cannot define a QFT model by its perturbative expansion. (Strocchi 2004, 501-502)

Thus, the fact that this model is not entirely realistic is not a striking blow against it since there is no realistic non-perturbative and non-trivial model in four spacetime dimensions in QFT; a fact which has not hampered other philosophers of physics from drawing conclusions about QFT.

### 3.9.3 Formal Underdetermination Objection

It could also be argued that once a representation of the CCRs is chosen that all UIRs constructed relative to it will have the same structure and myriotic properties. The underdetermination of choosing a representation is only formal and not substantial since all UIRs will have the same myriotic properties relative to the original representation. Thus, there is no ontological difference in choosing one UIR over another. This argument has two difficulties. (1) In the number occupation distribution discussed in section three, a UIR can be constructed relative to any representation if the sequences belonging to their equivalence classes differ in an infinite number of places. A UIR will thus have an infinite number of particles and there will be an infinite number of different states occupied. For example, the *a*-representation will only have a finite number of particles, so its equivalence class of sequences will only contain sequences with a finite number of non-zero entries. A $UIR_b$ relative to the *a*-representation will have an equivalence class of sequences where each sequence differs from the sequences in the *a*-representation's equivalence class of sequences in an infinite number of places. The $UIR_b$ will have an infinite number of particles. If a different $UIR_{b'}$ is chosen that is unitarily inequivalent to the *a*-representation as well as $UIR_b$, then the sequences in the equivalence class of $UIR_{b'}$ will differ from the sequences in the equivalence class of $UIR_b$ in an infinite number of places. Thus, while both $UIR_b$ and $UIR_{b'}$ will have an infinite number of particles, their particles will be in an infinite number of different states relative to each other! The fact that the states of their particles have an infinite number of differences can hardly be considered a formal or non-substantial difference. As a concrete example, consider an infinite spin lattice. If the electrons at each lattice site are compared for two UIRs, there will be an infinite number of lattice sites where the electrons have different spins (see section 2.3 of (Sewell 1986)). (2) Different UIRs will be useful for different physical situations. $UIR_b$ and $UIR_{b'}$ could correspond to different coupling constants or different masses. Once these parameters are fixed, representations with the incorrect coupling constant or mass will be eliminated as possible representations for that experiment. Thus, one cannot choose any UIR lying around for a particular experiment.

### 3.10 CONCLUSIONS

The ease with which UIRs can be built in QFT was shown above using a simple model. It was also shown how other UIRs may be constructed using a diverse set $\{c(k), g, t, m\}$ of parameters that includes the coupling constant, time, and mass. Other transformations can be constructed that involve a greater degree of complexity, such as the Bogoliubov transformation, but it turns out that the procedure that is used in doing so and the consequences that follow from it are not very different from those considered above.

Moreover, the appearance of UIRs is a direct consequence of a key physical requirement: invariance of the vacuum under spatial translations. This is what gives Haag's theorem its surprising and powerful force; namely, that there is no proper unitary transformation that will model a free system evolving into an interacting system. The Fock representation is useful for free quantum fields. By contrast, interacting quantum fields require representations that are unitarily inequivalent to the Fock representation (unless the requirement that the vacuum be invariant under spatial translations is eliminated, which does not really seem to be a viable option). Thus, if we want a QFT that is minimally consistent with the properties of the vacuum and models interactions, then one must use UIRs.

UIRs also dramatically impact the nature of quantum fields. There is a continuum of different ways to build a representation of a quantum field; some of them have a particle interpretation while others do not. Further, there is a continuum of different vacuum states, only one of which is the *bare* vacuum state. No one representation will suffice in general to characterize an interacting field over a period of time; a continuum of UIRs will be required. As a result of this embarrassment of riches, the ontology of the quantum field and the vacuum is decidedly non-unique.

## 3.11 APPENDIX

To show that a unitary operator does in fact transform one creation / annihilation operator set into another it is very helpful to use the following operator identity.

$$e^{A} B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \dots$$

Using this formula, one can show that there is a unitary operator $U_{ab}$ that transforms $a(\boldsymbol{k})$ into $b(\boldsymbol{k})$ (i.e., $b(\boldsymbol{k}) = U_{ab} a(\boldsymbol{k}) U_{ab}^{-1}$); namely, $U_{ab} = e^{A}$, where $A = \int d^3\boldsymbol{j}\ \left(c^*(\boldsymbol{j})a(\boldsymbol{j}) - c(\boldsymbol{j})a^\dagger(\boldsymbol{j})\right)$. The commutator

$$[A, a(\boldsymbol{k})] = \int d^3\boldsymbol{j}\ \left(c^*(\boldsymbol{j})a(\boldsymbol{j})a(\boldsymbol{k}) - c(\boldsymbol{j})a^\dagger(\boldsymbol{j})a(\boldsymbol{k}) - c^*(\boldsymbol{j})a(\boldsymbol{k})a(\boldsymbol{j}) + c(\boldsymbol{j})a(\boldsymbol{k})a^\dagger(\boldsymbol{j})\right)$$

$$= \int d^3\boldsymbol{j}\ \left(c(\boldsymbol{j})[a(\boldsymbol{k}), a^\dagger(\boldsymbol{j})] - c^*(\boldsymbol{j})[a(\boldsymbol{k}), a(\boldsymbol{j})]\right)$$

$$= \int d^3\boldsymbol{j}\ \left(c(\boldsymbol{j})\delta(\boldsymbol{k} - \boldsymbol{j}) + 0\right) = c(\boldsymbol{k}),$$

where the last step uses the CCRs at (3.3.1). Since $\left[A, a(\boldsymbol{k})\right] = c(\boldsymbol{k})$, the higher order terms are zero since the commutator of an operator with a complex number is zero, i.e., $\left[A, \left[A, a(\boldsymbol{k})\right]\right] = \left[A, c(\boldsymbol{k})\right] = 0$. Therefore,

$$U_{ab} a(\boldsymbol{k}) U_{ab}^{-1} = a(\boldsymbol{k}) + \left[A, a(\boldsymbol{k})\right] + \frac{1}{2!}\left[A, \left[A, a(\boldsymbol{k})\right]\right] + \ldots = a(\boldsymbol{k}) + c(\boldsymbol{k}) + 0 + \ldots = b(\boldsymbol{k}).$$

# Chapter Four

## 4.1 INTRODUCTION

The last chapter examined UIRs in canonical QFT and provided a model to "see" how unitary inequivalence can be generated in a variety of settings as well as its impact on dynamics. However, while canonical QFT is still the starting point for many QFT textbooks, it is not very mathematically rigorous. This does not bother many physicists now nor during the birth of QFT. The attitudes of most leading theoretical physicists from the 1940s was, "What we are doing is too important to be diverted by any demands of mathematical rigor…Calculate and see what turns up."[52] (Wightman 1996, 173-174) The impressive success of perturbative renormalized QFT encouraged this attitude. However, pertubative renormalized QFT did not address the issues of mathematical rigor or how to non-perturbatively formulate the theory. Friedrichs (discussed in (Haag 1996, 53)) compared his feelings about the QFT literature at this time to an archeologist finding the writings of an advanced civilization written in strange symbols; they were obviously written by intelligent people but what were they saying? Renormalization was a complicated set of procedures that required the familiarity that only comes from doing many calculations. However, the models for weak

[52] Wightman (1996) tells a story about von Neumann attending a lecture by Schwinger in the 1940s at the Institute for Advanced Study in Princeton. Schwinger claimed a certain operator was unitary and then discovered that it did not have an inverse. Von Neumann pointed out that in Hilbert space a unitary operator always has an inverse. Schwinger's response was along the lines that he would decide the meaning of "unitary."

interactions were not renormalizable and perturbation expansions were not useful for meson theories of the strong interaction. These problems along with the work of van Hove, Friedrichs, and Haag inspired mathematically inclined physicists in the late 1950s to search for a solid mathematical foundation for non-pertubative relativistic QFT. The efforts of people such as Wightman, Gårding, Haag, and Segal are often classified as *axiomatic quantum field theory*, though Wightman and Gårding had a very different approach than Haag and Segal. Wightman and Gårding were inspired by Schwartz's theory of distributions and developed an axiomatic version of relativistic QFT in the late 1950s where quantum fields were understood as operator-valued distributions satisfying conditions such as local commutativity and asymptotic completeness.[53] While the Wightman-Gårding approach deserves further study, it will be postponed for future research. Haag and Segal went a different way and built on the operator algebra work of von Neumann, Gelfand and Naimark, and Segal in the 1940s. Both approaches had to deal with the issue of UIRs.[54] The algebraic approach pioneered by Segal, Haag, Araki, and Kastler (to name just a few) is still an active area of research for mathematically-minded physicists, physically-minded mathematicians, and philosophers of physics. The only in-depth discussions of UIRs in the philosophy literature use the algebraic approach.

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53 For more information about Wightman's axiomatic formulation of QFT, see (Streater and Wightman 2000).

54 The "practically-minded" physicists dealt with UIRs by introducing cut-offs, so that the system only has a finite number of degrees of freedom, quantizing with the Fock representation, and then studying the system as the limits are removed. Of course, one could use a non-Fock

According to Wightman (1989, 624), Haag was never comfortable with the technicalities of dealing with unbounded operators and their domains, which is most likely one of the reasons Haag pursued his own version of QFT instead of Wightman and Gårding's.[55] Haag's algebraic approach used algebras of bounded operators generated by observables in open regions of spacetime. Since these open regions of spacetime are bounded and only have bounded operators defined on them, no domain problems arise. The issue of which type of algebra to pick had not reached a consensus by most mathematical physicists in the late 1950s to early 1960s. Segal (1957) advocated using C*-algebras while Haag (1957) (Haag and Schroer 1962) and Araki (1962) were using von Neumann algebras. Haag and Araki continued von Neumann's idea to use "rings of operators" (later called von Neumann algebras) as the preferred mathematical structure for QFT. Segal strongly criticized the use of von Neumann algebras as the appropriate algebra for QFT.

> It may also be helpful to compare our approach with one sketched by von Neumann, in which field dynamics is likewise to be expressed more or less in terms of automorphisms of an algebra. Apart from this similarity, there appears to be nothing in common between the approaches. The elegant and somewhat formal idea of von Neumann is that all the measurable field-theoretic variables should be expressible in terms of a 'type $II_1$' ring, whose automorphisms are expressible by unitary operators, which however are in general outside the ring; it is based technically on a *weakly* closed ring. The present intuitive idea is roughly that the only measurable field-theoretic variables are those that can be expressed in terms of a *finite* number of canonical operators, or *uniformly* approximated by such; the technical basis is a *uniformly* closed ring (more exactly, an abstract C*-algebra). The crucial difference between the two varieties of

representation to quantize the system, but as Wightman sarcastically remarked, "there is no other choice possible!" (Wightman 1989)

55 Haag did briefly use local algebras of unbounded operators (see (Haag 1957)).

> approximation arises from the fact that, in general, weak approximation has only analytical significance, while uniform approximation may be defined operationally, two observables being close if the maximum (spectral) value of their difference is small. More technically, weak approximation depends on the particular representation of the canonical operators, and also will be affected by an enlargement of the physical system under consideration, while uniform approximation is independent of the particular representation of the canonical variables, and is unaffected by enlargement of the system. The weak closure in analytically relevant concrete representations (e.g., the zero-interaction representation) of the present algebra of field observables may well consist of all bounded operators, and so have little connection with a ring of type $II_1$. (Segal 1959, 6-7)

Segal's argument for why abstract C*-algebras should be preferred to von Neumann algebras is based on his assertion that the uniform topology (also called the norm topology) has an operational definition and hence should be preferred to von Neumann algebras which are dependent on the topology of the particular Hilbert space on which each von Neumann algebra is defined. The reason Segal claimed that von Neumann algebras have only analytical significance is that once a representation is made of a C*-algebra on a Hilbert space it can be closed in the weak operator topology and thus become a von Neumann algebra. Thus, many – but not all – of the operators defined in the von Neumann algebra have abstract counterparts in the C*-algebra. I will return to this point later in the chapter.

The abundance of UIRs made it unclear which representation to use to model a particular situation. This is called the representation problem. Segal thought that the abstract nature of C*-algebras eliminated the representation problem since UIRs only appear once Hilbert spaces are used. In place of unitary operators, Segal (1959) thought that automorphisms of the C*-algebra could describe the dynamics of QFT. Some of these automorphisms would not be implementable by unitary operators. That is consistent with Haag's theorem, which showed the problems involved in modeling a free system evolving into an interacting system in QFT by unitary operators. Haag and Araki eventually accepted C*-algebras as the appropriate algebra for QFT. The pioneering paper by Haag and Kastler (1964) put forward a C*-algebraic axiomatic formulation of QFT. That framework is more commonly known as *algebraic quantum field theory* (AQFT), though many of its proponents prefer to call it *local quantum physics*. Haag (1996, 111) did not accept Segal's solution to the representation problem because it lacked a physical interpretation and Segal's proposal that the S-matrix be considered an automorphism of the algebra was "unacceptable." Kastler found an algebraic "solution" to the representation problem where the axioms Haag had been using were satisfied.

> After Rudolf had invited me to spend a year in Urbana, he confronted me with several *a priori* unrelated insights, one of them based on the postulate that King Solomon could not decide between two physicists working with "physically equivalent representations" of the same C*-algebra. After months of inconclusive investigations of his claims, I had the luck of finding a theorem of Fell in the bibliography of Guichardet's thesis (which I had providentially taken with me) verifying all of Rudolf's prophecies. The resulting coherence of vision led us to write an article on "An algebraic approach to quantum field theory" which was a hit, perhaps because it seemed to propose a new way of combining physics and mathematics. This paper formulated an axiomatic foundation for the net of local algebras.[56] (Kastler 2003, 4-5)

[56] Fell's theorem, which established the technical notion of *weak equivalence* upon which Haag and Kastler's concept of *physical equivalence* is based, will be discussed in detail in the next section.

Haag and Kastler's (1964) "solution" to the problem of UIRs was supoosed to eliminate the issue of having to choose a representation. This chapter will provide a complete analysis of this "solution."

The choice of C*-algebras was very fruitful in both AQFT and algebraic quantum statistical mechanics (AQSM). For philosophers of physics, the algebraic approach is appealing for its conceptual clarity and the ability to formulate very precise questions and construct proofs to answer these questions. The algebraic framework is also very flexible; it can be used to formulate quantum mechanics, AQFT, AQSM, and AQFT on curved spacetime. It exhibits the essential structural details of the preceding physical theories using powerful and rigorous mathematical results. Of particular interest here is Haag and Kastler's proposed solution to the representation problem and its connection with physical equivalence, i.e. operational equivalence, of UIRs. The influence of this solution has led some of the people working in the algebraic approach to claim that all of the physical content of QFT is contained in the abstract C*-algebra.

The basic concepts and axioms of the algebraic approach are collected in section 4.2. Section 4.3 will examine the mathematical notion of weak equivalence, while section 4.4 will explain Haag and Kastler's notion of physical equivalence how it relates to weak equivalence. The philosophical consequences of Haag and Kastler's notion of physical equivalence will be discussed in section 4.5. The connections between three different types of equivalence (unitary, quasi, and weak (physical)) for representations of both C*-

algebras and W*-algebras will be explained in section 4.6. Three special types of representations will be examined in section 4.7 and it will be shown how the mere unitary inequivalence of two concrete C*-algebras entails that Haag and Kastler's notion of physical equivalence is violated for their associated concrete W*- algebras. While it will be shown that physical equivalence does not eliminate the physical significance of UIRs, it can be used as a condition for when two representations are *classically equivalent*, which is discussed in section 4.8. This raises the question of what algebraic structure contains classical observables. The options are examined in section 4.9. In section 4.10, I provide a method of building classical observables that uses UIRs. The global nature of classical observables is discussed in section 4.11. An application of the previous results is given for the Unruh effect in section 4.12. Conclusions are provided in section 4.13. If a theorem has no proof in this chapter, then it has been proved elsewhere and the reader should consult the citation for the proof. All theorems with proofs are original. If one of my proofs only takes a few sentences, then it is kept in the text. Longer proofs are provided in section 4.14.

## 4.2 INTRODUCTION TO THE ALGEBRAIC APPROACH

### 4.2.1 Algebraic Concepts

For philosophers of physics who are only familiar with Hilbert spaces and non-relativistic quantum mechanics, the algebraic approach appears complicated and strange at first glance. This section has the modest goal of introducing the key concepts and mathematical structures in a way that makes contact with notions more familiar to philosophers of physics: vector spaces and Hilbert spaces. The following diagram will be the key for unlocking the mysteries of the algebraic approach in this dissertation.

C*-Algebra W*-Algebra

$$
\begin{tikzcd}
\text{Abstract:} & \mathcal{A} \arrow[r, "*"] \arrow[d, "\pi_\omega"'] & \mathcal{A}^* \arrow[r, "*"] & \mathcal{A}^{**} \arrow[d, "\pi^w_{\tilde{\omega}}"'] \\
& & \begin{matrix} \cup \\ \omega \in \mathcal{A}_1^{*+} \end{matrix} & \\
\text{Concrete:} & \pi_\omega(\mathcal{A}) \arrow[rr, "{''}\ \text{or}\ -"] & & \pi_\omega(\mathcal{A})'' = \pi^w_{\tilde{\omega}}\left(\mathcal{A}^{**}\right) = \pi_\omega(\mathcal{A})^-
\end{tikzcd}
$$

Figure 4.1: Overview of the Algebraic Structures

The first thing to notice is that there is an abstract and a concrete level in the algebraic approach. On a first pass, one can think of the concrete level as involving operators defined on Hilbert spaces and the abstract level as where operators are not defined on a Hilbert space.

Beginning in the upper lefthand corner of the diagram above, there is an abstract C*-algebra $\mathcal{A}$. An algebra is essentially a set of elements that is closed under linear combinations and products. It is a mathematical structure that has more structure than the more familiar notion of a vector space.

*V* is a *vector space* over the complex numbers $\mathbb{C}$ if the following properties hold, where $\alpha, \beta \in \mathbb{C}$ (the complex numbers) and $A, B \in V$.

1. *V* is an Abelian group with respect to (+), i.e.
   a. If $A, B \in V$, then $A + B \in V$. (Closure with respect to +)
   b. If $A, B, C \in V$, then $A + (B + C) = (A + B) + C$. (Addition associativity)
   c. There exists an $E \in V$ such that for all $A \in V$, $A + E = E + A = A$. (Existence of identity element)
   d. For every $A \in V$, there exists an element $F \in V$ such that, $A + F = F + A = E$. (Existence of inverse elements)
   e. For all $A, B \in V$, $A + B = B + A$. (Abelian property, also known as commutativity of addition)
2. *V* is closed with respect to its multiplication $(\cdot)$ operation applied to a complex number $\alpha \in \mathbb{C}$ and an element $A \in V$: $\alpha \cdot A \in V$.
3. $\alpha \cdot (\beta \cdot A) = (\alpha\beta) \cdot A$
   (Multiplicative associativity with respect to complex numbers)
4. $(\alpha + \beta) \cdot A = \alpha \cdot A + \beta \cdot A$ and $\alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B$
   (Multiplication is distributive with respect to complex numbers)
5. There exists a complex number, denoted by 1, such that for all $A \in V$, $1 \cdot A = A$
6. For all $A \in V$, $0 \cdot A = E$.

An algebra is essentially a complex vector space such that it is closed with respect to the multiplication operation $(\bullet)$ (i.e., if $A, B \in V$, then $A \bullet B \in V$) and it is associative with respect to its elements and the complex numbers.[57] A *-algebra adds a conjugation mapping $\left(^{*}\right)$ defined below. A normed algebra adds a norm $\|.\|$ to the algebra. Finally, a Banach algebra is a normed algebra which is complete with respect to its norm. Completeness is the requirement that every Cauchy sequence of elements of the algebra converges to an element in the algebra. A C*-algebra is a Banach *-algebra whose norm satisfies the following additional requirement (e).

A *C*-algebra* $\mathcal{A}$ is a vector space over the field of complex numbers $\mathbb{C}$ with the following algebraic features, (a) and (b), and topological features, (c), (d), and (e):

(a) A multiplication mapping from $\mathcal{A}$ into $\mathcal{A}$ that satisfies these three conditions for all $A,B,C \in \mathcal{A}$ and $\lambda \in \mathbb{C}$: $A(B+C) = AB + AC$, $A(BC) = (AB)C$, $A(\lambda B) = \lambda(AB)$.

(b) An involution which is a mapping $^{*}$ from $\mathcal{A}$ into $\mathcal{A}$ that satisfies these three conditions for all $A,B \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$: $(A^{*})^{*} = A$, $(AB)^{*} = B^{*}A^{*}$, $(\lambda A + \mu B)^{*} = \overline{\lambda} A^{*} + \overline{\mu} B^{*}$. $\overline{\lambda}$ and $\overline{\mu}$ are the complex conjugates of $\lambda$ and $\mu$.

(c) A norm $\|.\|$ that satisfies $\|AB\| \leq \|A\|\|B\|$ for all $A,B \in \mathcal{A}$.

(d) Completeness with respect to the norm topology, the topology given by the metric induced by the norm.

(e) $\|A^{*}A\| = \|A\|^{2}$

[57] For notational simplicity, this symbol is not used in what follows when two elements *A* and *B* of an algebra are multiplied, e.g. *AB*.

The dual of $\mathcal{A}$, denoted by $\mathcal{A}^*$, is the complete set of bounded linear functionals on $\mathcal{A}$.[58] A linear functional $\omega$ is a mapping from $\mathcal{A}$ to the complex numbers ($\omega : \mathcal{A} \to \mathbb{C}$) such that $\omega(\alpha A + \beta B) = \alpha\omega(A) + \beta\omega(B)$ where $\alpha, \beta \in \mathbb{C}$ and $A, B \in \mathcal{A}$. It can be shown that $\mathcal{A}^*$ is a Banach space.[59] A *state* is a linear functional that is positive ($\omega(A^* A) \geq 0$ for all $A \in \mathcal{A}$) and normed ($\omega(I) = 1$). The complete set of states in $\mathcal{A}^*$ is denoted as $\mathcal{A}_1^{*+}$. Returning to the diagram and staying at the abstract level, a move to the right by taking the dual of the C*-algebra $\mathcal{A}$ generates $\mathcal{A}^*$ within which the states $\mathcal{A}_1^{*+}$ of $\mathcal{A}$ reside, i.e. $\mathcal{A}_1^{*+} \subset \mathcal{A}^*$. An abstract state $\varphi$ is called *mixed* if it can be written as a convex combination $\varphi = \sum_{i=1}^{n} \lambda_i \psi_i$ of abstract states $\psi_i$, where $\lambda_i > 0$, $\varphi \neq \psi_i$, and $\sum_{i=1}^{n} \lambda_i = 1$. If $\varphi$ cannot be written as a convex combination of states, it is called a *pure state*. A C*-algebra is called a W*-algebra if it is the dual space of a Banach space. Since $\mathcal{A}^*$ is a Banach space, taking the dual of $\mathcal{A}^*$ generates a W*-algebra $\mathcal{A}^{**}$ that is called the *bidual*. It is rather surprising that $\mathcal{A}^{**}$ has algebraic structure.

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58 Unfortunately, the symbol for taking the dual is the same as the symbol for an involution in the literature. The dual is the set of all bounded linear forms over $\mathcal{A}$ while conjugation is a mapping from the algebra to the same algebra.

59 A Banach space is a normed vector space that is complete with respect to the metric induced by the norm.

A representation of a C*-algebra $\mathcal{A}$ is a map $\pi_\omega$ from $\mathcal{A}$ into the set of bounded operators $\mathcal{B}(\mathcal{H}_{\pi_\omega})$ of an associated Hilbert space $\mathcal{H}_{\pi_\omega}$ that preserves the algebraic relations between the elements of $\mathcal{A}$. The resulting concrete C*-algebra, the image of $\mathcal{A}$ in $\mathcal{H}_{\pi_\omega}$, is denoted as $\pi_\omega(\mathcal{A})$. To say that $\pi_\omega$ preserves the algebraic relations between the elements of $\mathcal{A}$ means that the following conditions are satisfied for any $A,B \in \mathcal{A}$ and $\alpha,\beta \in \mathbb{C}$:

$$\pi_\omega(\alpha A + \beta B) = \alpha\pi_\omega(A) + \beta\pi_\omega(B)\ ,$$
$$\pi_\omega(AB) = \pi_\omega(A)\pi_\omega(B)\ \text{, and}$$
$$\pi_\omega(A^*) = \pi_\omega(A)^*\ .$$

$\pi_\omega(A)$ denotes the concrete operator in $\mathcal{H}_{\pi_\omega}$ that corresponds to the abstract operator $A \in \mathcal{A}$. In the diagram, beginning with $\mathcal{A}$ in the upper lefthand corner, a representation $\pi_\omega$ can be built by picking an abstract state $\omega$ in $\mathcal{A}_1^{*+}$. $\pi_\omega(\mathcal{A})$ is a *concrete* C*-algebra and $\pi_\omega$ is a *C*-representation*.

It turns out that given an abstract state $\omega$ a Hilbert space can be constructed from it. This should not be too surprising since $\mathcal{A}$ is already a vector space. One of the key structures missing from $\mathcal{A}$ is an inner product.

A (complex) *pre-Hilbert space* is a complex vector space $\mathcal{H}$ with an inner product $\langle\ |\ \rangle$. The inner product is a map $\mathcal{H}\times\mathcal{H}\to\mathbb{C}$ such that the following properties hold.

(IP1) $\langle x|\alpha y + \beta z\rangle = \alpha\langle x|y\rangle + \beta\langle x|z\rangle$, $x,y,z \in \mathcal{H}$ and $\alpha,\beta \in \mathbb{C}$

(IP2) $\langle x|y\rangle = \overline{\langle y|x\rangle}$

(IP3) $\langle x | x \rangle \geq 0$

(IP4) $\langle x | x \rangle = 0$ if and only if $x = 0$

The last property allows a norm for the Hilbert space to be defined: $\|x\|^2 = \langle x | x \rangle$. An inner product is complete if and only if every Cauchy sequence in $\mathcal{H}$ converges in norm to some vector in $\mathcal{H}$. A pre-Hilbert space $\mathcal{H}$ that is complete with respect to the norm is called a *Hilbert space*.

The standard procedure for generating Hilbert spaces from a C*-algebra $\mathcal{A}$ is called the *GNS construction* (so called since Gelfand, Naimark, and Segal first formulated it). They proved that for every $\omega \in \mathcal{A}_1^{*+}$, one may construct a representation $\pi_\omega$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\pi_\omega}$ in such a way that there is a *cyclic*[60] vector $\Omega_{\pi_\omega} \in \mathcal{H}_{\pi_\omega}$ that satisfies $\omega(A) = \langle \Omega_{\pi_\omega} | \pi_\omega(A) | \Omega_{\pi_\omega} \rangle$ for all $A \in \mathcal{A}$. Each triple $\langle \mathcal{H}_{\pi_\omega}, \pi_\omega, \Omega_{\pi_\omega} \rangle$ is unique up to unitary equivalence. The idea behind the proof of GNS is that an abstract state acting on two operators in $\mathcal{A}$ is similar to an inner product. They both map two elements to the complex numbers. This abstract state does not satisfy (IP4). However, $\mathcal{A}$ can be quotiented out by elements of $\mathcal{A}$ that do not satisfy (IP4). This abstract state will then satisfy (IP1-4) and is a positive-definite inner product. The inner product defines a norm and

[60] A vector $\Omega_{\pi_\omega} \in \mathcal{H}_{\pi_\omega}$ is cyclic if $\{\pi_\omega(A) | \Omega_{\pi_\omega} \rangle | A \in \mathcal{A}\}$ is dense in $\mathcal{H}_{\pi_\omega}$.

a pre-Hilbert space has been constructed. By completing the pre-Hilbert space with respect to this norm, a Hilbert space can be built. A GNS-type proof can be constructed for other types of abstract algebras as long as the algebra has an involution operation and states can be defined on it. Thus, as long as the algebra is a *-algebra, a Hilbert space and representation can be constructed.

There is also a *reverse GNS process* where states in a Hilbert space have abstract counterparts in $\mathcal{A}_1^{*+}$. For each Hilbert space $\mathcal{H}_{\pi_\omega}$ associated with a representation $\pi_\omega$ there will be a collection of vector states and density matrices defined on $\mathcal{H}_{\pi_\omega}$. For each vector state $\left|\Psi_{\pi_\omega}\right\rangle$ and density matrix $\rho_{\pi_\omega}$ there is a corresponding abstract state in $\mathcal{A}_1^{*+}$ ($\Psi_\omega$ and $\rho_\omega$, respectively) defined by the following relations.

$$\Psi_\omega(\mathsf{A}) = \left\langle \Psi_{\pi_\omega} \middle| \pi_\omega(\mathsf{A}) \middle| \Psi_{\pi_\omega} \right\rangle$$
$$\rho_\omega(\mathsf{A}) = \mathrm{Tr}\, \rho_{\pi_\omega}\, \pi_\omega(\mathsf{A})$$

In other words, for each concrete state (vector state or density operator) in a Hilbert space there corresponds an abstract counterpart in $\mathcal{A}_1^{*+}$. Every density operator in $\mathcal{H}_{\pi_\omega}$ has an abstract counterpart in $\mathcal{A}_1^{*+}$. The set of all abstract states that are density operators in $\mathcal{H}_{\pi_\omega}$ is a norm closed convex subset of $\mathcal{A}_1^{*+}$ called the folium $\mathfrak{F}_\omega$.

Staying at the concrete C*-level, there are two equivalent ways to move to the right and construct a von Neumann algebra from a concrete C*-algebra

$\pi_\omega(\mathcal{A})$: (1) take the bicommutant[61] of $\pi_\omega(\mathcal{A})$, i.e. $\pi_\omega(\mathcal{A})''$, or (2) close $\pi_\omega(\mathcal{A})$ in the weak operator topology[62], i.e. $\pi_\omega(\mathcal{A})^-$. This is one of the distinguishing features of von Neumann algebras: they can be generated topologically using the weak operator topology or algebraically using the bicommutant. One crucial difference between $\pi_\omega(\mathcal{A})$ and $\pi_\omega(\mathcal{A})''$ is the difference in the topology defined on $\mathcal{B}(\mathcal{H}_{\pi_\omega})$. A Hilbert space can sustain many different topologies. In the case of $\pi_\omega(\mathcal{A})$, it sustains the norm topology which is fairly strict, i.e. there are some sequences $\pi_\omega(A_i) \to \pi_\omega(A)$ that will not converge in the norm topology. However, in the case of $\pi_\omega(\mathcal{A})''$, $\mathcal{B}(\mathcal{H}_{\pi_\omega})$ sustains the weak operator topology. The norm topology is stronger, or more fine, than the weak operator topology, which is a weaker or more coarse topology. In other words, every sequence $\pi_\omega(A_i) \to \pi_\omega(A)$ that converges in the norm topology also converges in the weak operator topology $\pi_\omega(A_i)'' \to \pi_\omega(A)''$ but a sequence that converges in the weak topology $\pi_\omega(B_i)'' \to \pi_\omega(B)''$ may not converge in the norm topology $\pi_\omega(B_i)$ to an element in $\pi_\omega(\mathcal{A})$. Since more limit points are being added when a

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[61] Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. $\mathcal{R}$'s *commutant* is defined as $\mathcal{R}' = \{B \in \mathcal{B}(\mathcal{H}) | AB = BA, \text{ for all } A \in \mathcal{A}\}$, and $\mathcal{R}'$'s bicommutant is $\mathcal{R}'' = (\mathcal{R}')'$. By definition, $\mathcal{R}$ is a *von Neumann algebra* if and only if $\mathcal{R} = \mathcal{R}''$.

[62] To say that a subset $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is *weakly closed* (i.e., that $\mathcal{A}$ is *closed in the weak operator topology*) means that any sequence $\{T_n\}$ of elements of $\mathcal{A}$ converges to another element $T \in \mathcal{R}$ in the sense that $\langle\Phi|T_n|\Psi\rangle \to \langle\Phi|T|\Psi\rangle$ for all $\Phi, \Psi \in \mathcal{H}$.

von Neumann algebra is created from a concrete C*-algebra, every von Neumann algebra is a concrete C*-algebra but not vice versa. Thus, when $\pi_\omega(\mathcal{A})$ is closed in the weak operator topology there are new observables which are limit points $\pi_\omega(A_i)'' \to \pi_\omega(A)''$. While each observable $\pi_\omega(\mathcal{A})$ has an abstract counterpart $A \in \mathcal{A}$, these new observables in $\pi_\omega(\mathcal{A})''$ have no abstract counterpart in $\mathcal{A}$. However, they will have an abstract counterpart in $\mathcal{A}^{**}$. There is also a natural embedding of $\mathcal{A}$ into $\mathcal{A}^{**}$ such that $\mathcal{A} \subseteq \mathcal{A}^{**}$.

The most common way to construct a von Neumann algebra is to start with a C*-algebra $\mathcal{A}$ and an abstract state $\omega$, construct a representation $(\mathcal{H}_{\pi_\omega}, \pi_\omega)$ via the GNS theorem, and then close $\pi_\omega(\mathcal{A})$ in the weak topology $\pi_\omega(\mathcal{A})''$.[63] However, there is an equivalent alternative way to build $\pi_\omega(\mathcal{A})''$ using the bidual $\mathcal{A}^{**}$. Since C*-algebras and W*-algebras are both *-algebras, the GNS theorem can be used to construct Hilbert spaces from both algebras. In order for $\omega$ to be a state for $\mathcal{A}^{**}$ it must be extended to be a normal state $\tilde{\omega}$ on $\mathcal{A}^{**}$.[64] If the GNS construction is done using $\tilde{\omega}$ and $\mathcal{A}^{**}$, then a von

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[63] Though the notation for a von Neumann algebra closed in the weak topology is $\pi_\omega(\mathcal{A})^-$, a von Neumann algebra is usually symbolized as $\pi_\omega(\mathcal{A})''$ even if it is being discussed as being closed in the weak topology. This is because $\pi_\omega(\mathcal{A})^- = \pi_\omega(\mathcal{A})''$.

[64] A linear functional $\tilde{\rho}$ on a W*-algebra $\mathcal{A}^{**}$ is said to be *normal* if and only if $\tilde{\rho}(\sup_\alpha T_\alpha) = \sup_\alpha \tilde{\rho}(T_\alpha)$ for every uniformly bounded increasing directed set $\{T_\alpha\}$ of positive elements of $\mathcal{A}^{**}$.

Neumann algebra $\pi_{\tilde{\omega}}^{w}\left(\mathcal{A}^{**}\right)$ is generated.[65] A von Neumann algebra is a concrete W*-algebra; it is a *W*-representation* of $\mathcal{A}^{**}$.[66] Thus, there is an isomorphism at the abstract level between the set of all states $\mathcal{A}_1^{*+}$ and the set of all normal states on $\mathcal{A}^{**}$ as well as an isomorphism between the set of representations on $\mathcal{A}$ and the set of W*-representations on $\mathcal{A}^{**}$. All representations $\left(\mathcal{H}_{\pi_\omega},\pi_\omega\right)$ of $\mathcal{A}$ can be uniquely extended $\left(\mathcal{H}_{\pi_\omega},\pi_{\tilde{\omega}}^{w}\right)$ to be a W*-representation of $\mathcal{A}^{**}$ and this extension is equal to the weak closure of $\pi_\omega\left(\mathcal{A}\right)$, i.e. $\pi_\omega\left(\mathcal{A}\right)'' = \pi_{\tilde{\omega}}^{w}\left(\mathcal{A}^{**}\right) = \pi_\omega\left(\mathcal{A}\right)^{-}$ (see theorem 4.2.7 (Bing-Ren 1992, 221-222)).

One special representation that will be used in this chapter and the next chapter is the *universal representation* $\pi_u\left(\mathcal{A}\right)$. Its von Neumann algebra is called the *universal enveloping von Neumann algebra* $\pi_u\left(\mathcal{A}\right)''$. The universal representation $\pi_u\left(\mathcal{A}\right)$ of the algebra $\mathcal{A}$ is a direct sum of representations using every state in $\mathcal{A}_1^{*+}$; i.e., it is defined as $\pi_u \equiv \bigoplus_{\rho\in\mathcal{A}_1^{*+}} \pi_\rho$ and is associated with the direct sum of Hilbert spaces $\mathcal{H}_{\pi_u} \equiv \bigoplus_{\rho\in\mathcal{A}_1^{*+}} \mathcal{H}_{\pi_\rho}$. By closing the universal representation in the weak operator topology, the universal enveloping von Neumann algebra $\pi_u\left(\mathcal{A}\right)''$ can be generated. Since every abstract state is used

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[65] For more details about representations of $\mathcal{A}^{**}$, see (Bing-Ren 1992, 221-222).

to construct the universal representation, which can then be closed in the weak operator topology, a large set of new observables is defined on $\mathcal{H}_{\pi_u}$. Each of these new observables will have an abstract counterpart in $\mathcal{A}^{**}$ since there is an isometric isomorphism between $\mathcal{A}^{**}$ and $\pi_u(\mathcal{A})''$ (for details see theorem 11 (Emch 1972, 121-122)). This indicates that $\mathcal{A}^{**}$ has a very large set of new observables and that, in general, the subset relation $\mathcal{A} \subseteq \mathcal{A}^{**}$ is proper: $\mathcal{A} \subset \mathcal{A}^{**}$.

### 4.2.2 Axioms for AQFT

In local AQFT, each open region $O$ of Minkowski spacetime $\mathcal{M}$ is associated with a set $\mathcal{A}(O)$ of elements of a C* algebra, the local observables in $O$. The regions $O \in \mathcal{M}$ are often taken to be double-cones, nonempty intersections of the interiors of a forward and a backward light cone. By taking the set theoretic union of all $\mathcal{A}(O)$ in $\mathcal{M}$ and closing in the norm topology, the algebra of all quasi-local observables can be defined: $\mathcal{A}_{\text{loc}} = \overline{\bigcup_{O \in \mathcal{M}} \mathcal{A}(O)}$. $\mathcal{A}_{\text{loc}}$ is assumed to satisfy a set fundamental physical conditions, known as "axioms of

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66 Roughly, a W*-representation is a *-homomorphism from an abstract W*-algebra to $\mathcal{B}(\mathcal{H}_{\pi_\omega})$ which is continuous in the topologies at the abstract and concrete levels. For the details, see (Bing-Ren 1992, 221).

local structure." Three key axioms of local structure are isotony, microcausality, covariance.[67]:

*Isotony*: If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$,

*Microcausality*: If $O_1$ and $O_2$ are spacelike separated, then $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$,

*Covariance*: If $g \in P_+^\uparrow$ and $A \in \mathcal{A}(O)$, then $\alpha_g[\mathcal{A}(O)] = \mathcal{A}(g[O])$.

Table 4.1: Axioms of AQFT

The idea behind isotony is that an observable measurable in $O_1$ is also measurable in a larger spacetime region $O_2$ containing $O_1$. Microcausality says that if two spacetime regions are spacelike separated, then any element from the algebra associated with one region must commute with any element from the algebra associated with the other. This is supposed to reflect the constraints imposed by relativity on spacetime. Covariance says that each element *g* of the restricted Poincaré group $P_+^\uparrow$ (one of four disjoint classes of the Poincaré group) may be represented as an automorphism $\alpha_g$ of the algebra $\mathcal{A}(O)$. These are the primary conditions, though additional conditions are often specified in various AQFT books and papers.

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[67] For more information on the axioms, see (Halvorson and Mueger 2006).

### 4.2.3 Axioms for AQSM

In quasi-local algebraic quantum statistical mechanics, the quasi-local C* algebra of observables is defined as follows. For each bounded subset $s \in \mathbb{R}^3$ ($\mathbb{R}^3$ is Euclidean space) and time $t \in \mathbb{R}$ there is an associated C*-algebra of observables $\mathcal{A}(s,t)$ whose elements are local observables in *s* at *t*. The quasi-local algebra at *t* is $\mathcal{A}_{\text{loc}}(t) = \overline{\bigcup_{s \in \mathbb{R}^3} \mathcal{A}(s,t)}$. The quasi-local algebra is

$\mathcal{A}_{\text{loc}} = \overline{\bigcup_{t \in \mathbb{R}} \mathcal{A}_{\text{loc}}(t)}$. It is assumed to satisfy certain fundamental physical conditions that are counterparts to the ones above for AQFT. Isotony is exactly analogous to that in AQFT, and it involves spatial regions instead of spacetime regions. Locality corresponds to commutativity of operators that are associated with disjoint spatial regions rather than spacelike separated regions of spacetime. Covariance is often associated with a continuous one parameter group of *-automorphisms of the C*-algebra of observables rather than the Poincaré group. With the mathematical details and the axioms of the algebraic approach in place, an evaluation of the effectiveness of weak (physical) equivalence can be begin.

## 4.3 Weak Equivalence

It is important to make a distinction between the notions of weak equivalence and physical equivalence. *Weak equivalence* is a mathematical concept that was introduced by Fell (1960, 375) while *physical equivalence*, which is discussed in the next section, is a semantic extrapolation of weak equivalence introduced by Haag and Kastler (1964). Before doing so, two key ideas must be introduced. First, a topology may be defined on the dual of the algebra, denoted by $\mathcal{A}^*$, where the abstract states are defined, that is called the weak*-topology on $\mathcal{A}^*$ (sometimes referred to as the $\sigma\left(\mathcal{A}^*,\mathcal{A}\right)$ topology) in which a basis of neighborhoods $N(\varphi,\{A_i\}_{i=1}^{k},\varepsilon)$ consists of all sets of the form:

$$\left\{\omega \in \mathcal{A}^* \middle| \left|\varphi\left(A_i\right) - \omega\left(A_i\right)\right| < \varepsilon\right\}$$

where $A_i \in \mathcal{A}$ and $\varepsilon > 0$. The weak*-neighborhoods of an abstract state $\varphi$ are indexed by a finite subset $A_i$ of $\mathcal{A}$ and a positive real number $\varepsilon$. Any continuous positive linear functional $\phi$ on $\mathcal{A}$ can be approximated in the weak*-topology by positive linear functionals of the form $\phi_n = \sum_{i=1}^{n}\left\langle \Psi_i \middle| A \middle| \Psi_i \right\rangle$, where $\|\phi_n\| \le \|\phi\|$. The second important concept is the *kernel* of a representation which is defined as the set of elements the representation maps to the zero operator in its Hilbert space: $\ker\pi \equiv \left\{A \middle| \pi\left(A\right) = 0, A \in \mathcal{A}\right\}$. The key theorem of Fell (1960, 367) can now be expressed as follows.[68]

[68] Fell used a family of representations in his equivalence theorem 1.2, but we do not need this additional complexity. Two of the four equivalent conditions from Fell's theorem 1.2 are not listed in our Fell's theorem and will not be used in what follows.

**Fell's Theorem:** Let $\mathcal{A}$ be a C*-algebra and $\pi_\varphi$ and $\pi_\psi$ be two representations of $\mathcal{A}$.[69] The following conditions are equivalent.

(1) the kernel of $\pi_\psi$ is contained in $\pi_\varphi$, i.e., $\ker \pi_\psi \subseteq \ker \pi_\varphi$
(2) every positive functional $\eta$ on $\mathcal{A}$ associated with $\pi_\varphi$ is a weak*-limit of finite sums $\gamma$ of positive functionals associated with $\pi_\psi$ for which $\|\eta\| \leq \|\gamma\|$.

If either of these conditions is satisfied, then $\pi_\varphi$ is *weakly contained* in $\pi_\psi$. Since the kernel of $\pi_\varphi$ is larger than $\pi_\psi$, $\pi_\varphi$ maps more elements of $\mathcal{A}$ to the zero element of $\mathcal{A}$, i.e., it is less faithful.[70] Fell's theorem is used to define *weak equivalence*.

**Definition:** Two representations $\pi_\varphi$ and $\pi_\psi$ of a C*-algebra $\mathcal{A}$ are *weakly equivalent* if and only if they are weakly contained in each other.

This allows Fell's theorem to be put in the following form.

**Fell's Theorem 2:** Two representations $\pi_\varphi$ and $\pi_\psi$ of a C*-algebra $\mathcal{A}$ are *weakly equivalent* if and only if $\ker \pi_\varphi = \ker \pi_\psi$.

The correspondence between abstract states and concrete states along with the imposition of the weak*-topology on $\mathcal{A}^*$ allows the vector states and density matrices of a Hilbert space $\mathcal{H}_{\pi_\varphi}$ to be approximated by states in a different Hilbert space $\mathcal{H}_{\pi_\psi}$ that may be associated with a UIR. Let $\mathfrak{S}_{\pi_\varphi}$ and $\mathfrak{S}_{\pi_\psi}$ be the set of all states on the concrete C*-algebras $\pi_\psi$ and $\pi_\varphi$ respectively.

[69] It is assumed throughout this chapter that all representations are nondegenerate, which can intuitively be thought of as assuming that the representations are not the zero representation.

Each element of $\mathfrak{S}_{\pi_\varphi}$ and $\mathfrak{S}_{\pi_\psi}$ has an abstract counterpart in $\mathcal{A}_1^{*+}$. They can thus be viewed as subsets of the set of abstract states in $\mathcal{A}_1^{*+}$; denoted by $\mathfrak{S}_\varphi$ and $\mathfrak{S}_\psi$.[71] The connection of this with Fell's theorem is given by Emch's (1972, 106) lemma where $\pi_\varphi$ is *weakly contained* in $\pi_\psi$.

> **Emch's Lemma:** Let $\mathcal{A}$ be a C*-algebra and $\pi_\varphi$ and $\pi_\psi$ be two representations of $\mathcal{A}$. The following three conditions are equivalent.
>
> (1) $\ker \pi_\psi \subseteq \ker \pi_\varphi$
> (2) Every state on $\mathcal{A}$ that is a vector state for $\pi_\varphi$ is the w*-limit of a net of finite convex combinations of states on $\mathcal{A}$ which are vector states associated with $\pi_\psi$.[72]
> (3) Every density matrix state associated with $\pi_\varphi$ can be approximated pointwise on $\mathcal{A}$, as close as we want by a density matrix associated $\pi_\psi$.
> (4) $\mathfrak{S}_\varphi \subseteq \mathfrak{S}_\psi$

In Emch's lemma on weak containment, condition (4) shows that if $\pi_\varphi$ is *weakly contained* in $\pi_\psi$, then the abstract states of $\pi_\varphi$ are a subset of the abstract states of $\pi_\psi$, i.e., $\mathfrak{S}_\varphi \subseteq \mathfrak{S}_\psi$. Each $\mathfrak{S}_{\pi_\omega}$ has vector states $\mathcal{V}_{\pi_\omega}$ and density operators

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[70] A representation is faithful if and only if its kernel is trivial, i.e. $\ker \pi \equiv \{0\}$ where $0 \in \mathcal{A}$.

[71] The general rule for my notation is that the states which are concrete states in a Hilbert space if there is a $\pi$ in the subscript. If there is no $\pi$ in the subscript, then the states are abstract states belonging to $\mathcal{A}_1^{*+}$.

[72] A net is a more generalized notion of sequence used to characterize convergence. For more details see (Emch 1972, 102). To say that $\phi$ is a convex sum of states $\phi_i$ means that $\phi = \sum_{i=1}^{n} \lambda_i \phi_i$, where $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

$\mathfrak{F}_{\pi_\omega}$.[73] Since every unit vector gives rise to a density operator, namely the projection onto that unit vector, the only abstract counterparts to the states in $\mathfrak{S}_{\pi_\omega}$ we need to consider are the algebraic states in $\mathcal{A}_1^{*+}$ which are expressible as density operators on $\pi_\omega(\mathcal{A})$. This set of abstract states is called the *folium* $\mathfrak{F}_\omega$. These relations are summarized below.

| | |
|---|---|
| Abstract Level: | $\mathcal{V}_\omega \subseteq \mathfrak{F}_\omega \subseteq \mathfrak{S}_\omega \subseteq \mathcal{A}_1^{*+}$ |
| Concrete Level: | $\mathcal{V}_{\pi_\omega} \subseteq \mathfrak{F}_{\pi_\omega} \subseteq \mathfrak{S}_{\pi_\omega}$ |

Table 4.2: Abstract and Concrete State Relationships

$\left(\mathcal{H}_{\pi_\varphi},\pi_\varphi\right)$ and $\left(\mathcal{H}_{\pi_\psi},\pi_\psi\right)$ are *weakly equivalent* if and only if for every density matrix $\rho_{\pi_\varphi}$ in $\mathfrak{F}_{\pi_\varphi}$ and weak*-neighborhood $N(\rho_\varphi,\{A_i\}_{i=1}^{k},\varepsilon)$ there exists a density matrix $\rho_{\pi_\psi}$ in $\mathfrak{F}_{\pi_\psi}$ such that $\rho_\psi \in N(\rho_\varphi,\{A_i\}_{i=1}^{k},\varepsilon)$. This implies that $\left(\mathcal{H}_{\pi_\varphi},\pi_\varphi\right)$ and $\left(\mathcal{H}_{\pi_\psi},\pi_\psi\right)$ are weakly equivalent if and only if the closure of the folium $\mathfrak{F}_\varphi$ in the weak*-topology on $\mathcal{A}^*$ is equal to the closure of the folium $\mathfrak{F}_\psi$ in the weak*-topology on $\mathcal{A}^*$, i.e., ${}^{w^*}\overline{\mathfrak{F}}_\varphi = {}^{w^*}\overline{\mathfrak{F}}_\psi$. This last equality will play an important role in the next chapter.

[73] $\mathfrak{S}_{\pi_\omega}$ can also have singular states.

In Fell's theorem 2, the notion of weak equivalence depends on the kernels of the two representations being equal. If one considers all nonzero *faithful* representations, i.e. representations that have trivial kernels, then they are all weakly equivalent to each other. This is how weak equivalence is usually presented in the literature. It is referred to as Fell's theorem even though this is not the way Fell stated his theorem.

> **Weak Equivalence:** All faithful representations of a C*-algebra are weakly equivalent.

If the C*-algebra is simple.[74], then all nonzero representations of it are faithful. Thus, each nonzero representation (i.e., $\pi_{\omega}(A) \neq 0$ for some $A \in \mathcal{A}$) is weakly equivalent to every other representation of a simple C*-algebra. Two physically important algebras, the Weyl CCR C*-algebra (Bratteli and Robinson 1997, 19-22) and the Weyl CAR C*-algebra (Bratteli and Robinson 1997, 15-16), are both simple.

## 4.4 Physical Equivalence

Haag and Kastler (1964) interpreted the mathematical concept of weak equivalence as a kind of *physical equivalence*. This notion was subsequently used by many people in AQFT to elevate the abstract C*-algebra to a position of physical importance while denigrating the significance of representations, in

[74] An algebra is simple if it has no closed two-sided ideals.

particular that of UIRs, using the following verificationist justification. Since only a finite number of experiments can be carried out and each experiment will only have finite accuracy, any state that has roughly the same expectation values as another state (within $\varepsilon$) for the same observables should be considered *physically equivalent*. In other words, if two states predict roughly the same values for a set of observables, then there is no physically significant difference between the two states. Given this interpretation of weak equivalence, it is easy to see how two weakly equivalent representations can be classified as satisfying this description of physical equivalence. In setting up an experiment or a series of experiments there are three important features:

(1) the state $\rho_{\pi_\varphi}$ of the system prior to measurement,
(2) the observable(s) $A_i$ that are going to be measured, and
(3) the accuracy of each measurement $\varepsilon_i$.

Let $\rho_{\pi_\varphi}$ be the density operator in $\left(\mathcal{H}_{\pi_\varphi}, \pi_\varphi\right)$ that is the state of the system prior to a measurement and $\rho_\varphi$ its abstract state counterpart in the folium $\mathfrak{F}_\varphi$. Once (2) and (3) have been fixed as well, there will be a weak*-neighborhood $N(\rho_\varphi, A_i, \varepsilon_i)$[75] around $\rho_\varphi$ in $\mathcal{A}_1^{*+}$ where $1 \le i \le k$. Let $\left(\mathcal{H}_{\pi_\psi}, \pi_\psi\right)$ be a UIR with respect to $\left(\mathcal{H}_{\pi_\varphi}, \pi_\varphi\right)$. Will there be an abstract state $\rho_\psi$ in the folium of $\mathfrak{F}_\psi$ that is physically equivalent in the above sense? If $\ker \pi_\varphi = \ker \pi_\psi$, then weak equivalence guarantees that such a state can be found in $\mathfrak{F}_\psi$, i.e., $\left| \rho_\varphi(A_i) - \rho_\psi(A_i) \right| < \varepsilon_i$. As mentioned above, if the algebra is simple and the representations are nonzero or both representations are faithful, then both representations are weakly equivalent and hence can be interpreted as physically equivalent.[76]

As long as faithful representations are constructed, the representations will be weakly (physically) equivalent. It is a relatively simple matter to construct faithful representations. Every C*-algebra has a faithful representation (theorem 4.5.6 of Kadison and Ringrose (1997a, 281)). If the representation is not faithful, then it can be made faithful. Since the kernel of any representation $\pi_\omega$ is a closed two-sided ideal, the C*-algebra $\mathcal{A}$ can be quotiented out, i.e., $\mathcal{A}/\ker \pi_\omega$, by the kernel of $\pi_\omega$ (see (Emch 1972, 79-80)). A representation of $\mathcal{A}/\ker \pi_\omega$ will automatically be faithful and it will be a concrete C*-algebra (Haag and Kastler 1964, 852). Given these mathematical facts and the semantic interpretation of physical equivalence, it is easy to understand why some algebraic quantum field

[75] In Fell's theorem, there is no subscript on the $\varepsilon$. To eliminate this lacuna, the value of $\varepsilon$ in the discussion of the weak*-topology on $\mathcal{A}^*$ can be considered the maximum experimental error, i.e., the largest of the $\varepsilon_i$.

[76] Of course, an obvious way that physical equivalence would be violated is if an infinite number of experiments was carried out. A more stringent algebraist could require that two representations are really physically equivalent if an abstract state can be found in the folium of the other representation such that the difference in expectation values for the states in the different foliums are only infinitesimally different for an infinite number of algebraic observables.

theorists claimed that all of the physical content in AQFT resides in the abstract C*-algebra $\mathcal{A}$.

### 4.5 PHILOSOPHICAL CONSEQUENCES OF WEAK EQUIVALENCE

Various phrases such as “algebraic imperialism” (Arageorgis 1995, 132) and “algebraic chauvinism” (Ruetsche 2003, 1334) have been used to describe the position of algebraic quantum field theorists in the 1960s.[77] Based on the notion of physical equivalence, it was claimed that Hilbert spaces and representations of a C*-algebra were dispensable and that the abstract C*-algebra contained all of the physical content of a theory.

> [T]he important thing here is that the observables form some algebra, and not the representation Hilbert space on which they act. (Segal 1967, 128)

> It is in this new notion of equivalence in field theory that the algebraic approach has its greatest justification. All the physical content of the theory is contained in the algebra itself; nothing of fundamental significance is added to a theory by its expression in a particular representation. From this point of view it becomes clear that only faithful representations are worth consideration because the existence of a non-trivial kernel for a representation implies that there are redundant elements in the algebra. (Robinson 1966, 488)

> The relevant object is the abstract algebra and not the representation. The selection of a particular (faithful) representation is a matter of convenience without physical implications. It may provide a more or less handy analytical apparatus. (Haag and Kastler 1964, 851-852)

> It [the concept of physical equivalence] shows indeed that the physically relevant object if not a concrete realization of $\mathcal{A}$ but the algebra itself, since any two different concrete realizations (i.e., faithful *-

[77] The details of this position and its use in the philosophical literature will be discussed in the next chapter.

> representations, or representations with zero kernel) will be physically equivalent. (Kastler 1964, 180)

Historically, there are two arguments that motivated this attitude. (1) In the 1940s-1960s, Segal advocated the position that the C*-algebra was the fundamental mathematical structure for quantum theory – not the Hilbert space. This was based on the GNS theorem which showed that Hilbert spaces could be constructed from the abstract C*-algebra. (2) Haag and Kastler's argument for physical equivalence was taken to show that there were no physically significant differences between any two UIRs. Given the lemmas and theorems associated with weak equivalence, the algebraic approach made a very strong case that UIRs had only formal significance.

However, Summers (2001, 145) has criticized the conclusion that physical equivalence entails that the choice of representation is a mere matter of convenience. Assuming that the representations chosen are faithful, three items must be fixed in order to use weak equivalence: (1) the state of the initial system, (2) a finite set of observables, and (3) the range of errors associated with each measurement. Summers argued that if there are any improvements to the experiment which reduce the errors or any changes to the experiment to include another set of observables, then the weak*-neighborhood will change and another approximate state will have to be found to reproduce the results of the new experiment. To be flexible enough to accommodate new experiments and experimental improvements one needs the correct state in the correct representation, which is not merely a matter of convenience.

Of course, any change in *any* of the three conditions would create a new weak*-neighborhood and depending on the nature of the changes the approximate state might or might not be different. But one of the algebraic imperialists could respond that when the experimental physicist sets up her lab and carries out her experiment that all three conditions would be fixed. Any change she makes could be easily compensated for by finding the new approximate state. However, this practical indistinguishability for states is only backwards-looking (Ruetsche 2007); that is, only after all three conditions are fixed can an approximate state be found. An operationalist should also have predictive instrumentality which is forward-looking and can predict which approximate state is appropriate. But a particularly ardent algebraic imperialist may view this as a minor inconvenience. A more telling critique of physical equivalence should proceed by showing that there are significant physical differences between UIRs and that the interpretation of weak equivalence as physical equivalence does not capture physicists' intuitive concept of physical equivalence.

### 4.6 Unitary, Quasi, and Weak Equivalence

Using weak equivalence to argue for the physical *insignificance* of UIRs would be particularly devastating if the only type of representation used in algebraic quantum physics is a representation of a C*-algebra. However, the most commonly used concrete algebra is a von Neumann algebra (see table 4.6

below) and hence W*-representations are more useful than C*-representations. If weak equivalence could be shown to hold for W*-representations as well, then a strong case could be made for the algebraic imperialist. It will be shown that the mere weak equivalence of C*-representations is not enough to guarantee the weak equivalence of their W*-representations.

The key notion involved in this analysis is quasi-equivalence. There are four equivalent definitions of quasi-equivalence which are summarized in the following theorem.

**Theorem 4.6.1:** Let $\mathcal{A}$ be a C*-algebra and $\pi_\varphi$ and $\pi_\psi$ be two representations of $\mathcal{A}$. If any of the following conditions is satisfied, then $\pi_\varphi$ and $\pi_\psi$ are quasi-equivalent.

(1) There is an isomorphism $\alpha : \pi_\varphi(\mathcal{A})'' \to \pi_\psi(\mathcal{A})''$ between the enveloping von Neumann algebras of $\pi_\varphi$ and $\pi_\psi$ such that $\alpha\left[\pi_\varphi(\mathcal{A})\right] = \pi_\psi(\mathcal{A})$ for any $A \in \mathcal{A}$.
(2) The foliums are equal, i.e. $\mathfrak{F}_\varphi = \mathfrak{F}_\psi$.
(3) Every subrepresentation of $\pi_\varphi$ is unitarily equivalent to some subrepresentation of $\pi_\psi$ and vice versa.
(4) Their enveloping central supports are equal, i.e. $C_{\tilde{\varphi}} = C_{\tilde{\psi}}$.[78]

[78] The enveloping central support $C_{\tilde{\varphi}}$ of a state $\tilde{\varphi}$ is defined as the smallest projection $P$ in the center of $\mathcal{A}^{**}$, denoted as $\mathfrak{Z}(\mathcal{A}^{**})$, such that $\tilde{\varphi}(P) = 1$, i.e., $C_{\tilde{\varphi}} := \inf\left\{P \in \mathfrak{Z}(\mathcal{A}^{**}) \middle| \tilde{\varphi}(P) = 1,\ P = P^* = P^2\right\}$. $\mathfrak{Z}(\mathcal{A}^{**})$ is the set of elements of $\mathcal{A}^{**}$ that commute with every element of $\mathcal{A}^{**}$; i.e. $\mathfrak{Z}\left(\mathcal{A}^{**}\right) \equiv \left\{A \in \mathcal{A}^{**} \middle| AB = BA, \text{ for all B} \in \mathcal{A}^{**}\right\}$.

Quasi-equivalence can be thought of roughly as unitary equivalence up to multiplicity. In general, quasi-equivalence is a weaker notion than unitary equivalence but it is stronger than weak (physical) equivalence.

Unitary Equivalence $\Rightarrow$ Quasi-Equivalence $\Rightarrow$ Weak (Physical) Equivalence

Table 4.3: Equivalence Relationships for C*-Representations

If $\pi_\varphi$ and $\pi_\psi$ are representations of a C*-algebra $\mathcal{A}$ and they are unitarily-equivalent , i.e., $\pi_\psi(\mathrm{A}) = U\pi_\varphi(\mathrm{A})U^{-1}$ for all $\mathrm{A} \in \mathcal{A}$, then their respective enveloping von Neumann algebras are unitarily equivalent, i.e., $\pi_\psi(\mathrm{A})'' = U\pi_\varphi(\mathrm{A})''U^{-1}$ (Kadison and Ringrose 1997b, 735). Weak (physical) equivalence requires that the kernels of the C*-representations $\pi_\varphi$ and $\pi_\psi$ be equal. The condition usually cited for two C*-representations $\pi_\varphi$ and $\pi_\psi$ to be weakly (physically) equivalent is that their kernels are equal to each other: $\ker \pi_\varphi = \ker \pi_\psi$. Since the notion of a kernel requires a representation, W*-representations also have kernels. Thus, the notion of weak (physical) equivalence can be extended to W*-representations of $\mathcal{A}^{**}$. Two W*-representations $\pi_{\tilde{\varphi}}^{w}$ and $\pi_{\tilde{\psi}}^{w}$ are weakly (physically) equivalent if and only if their kernels are equal to each other: $\ker \pi_{\tilde{\varphi}}^{w} = \ker \pi_{\tilde{\psi}}^{w}$.

Just as the unitary equivalence of two C*-representations implies that they are weakly equivalent, the unitary equivalence of two W*-representations implies that they are weakly (physically) equivalent. If $\mathrm{A} \in \ker \pi^{w}_{\tilde{\varphi}}$, then $\pi^{w}_{\tilde{\psi}}(\mathrm{A}) = U\pi^{w}_{\tilde{\varphi}}(\mathrm{A})U^{-1} = U0U^{-1} = 0$, so $\mathrm{A} \in \ker \pi^{w}_{\tilde{\psi}}$ and $\ker \pi^{w}_{\tilde{\varphi}} \subseteq \ker \pi^{w}_{\tilde{\psi}}$. A similar argument shows that $\ker \pi^{w}_{\tilde{\psi}} \subseteq \ker \pi^{w}_{\tilde{\varphi}}$, thus, $\ker \pi^{w}_{\tilde{\varphi}} = \ker \pi^{w}_{\tilde{\psi}}$ which implies that $\pi^{w}_{\tilde{\varphi}}$ and $\pi^{w}_{\tilde{\psi}}$ are weakly (physically) equivalent to each other. The connection between the weak equivalence of W*-representations and C*-algebras is given by the following theorem.

> **Theorem 4.6.2 (Emch 1972, 124):** Two representations $\pi_{\varphi}$ and $\pi_{\psi}$ of a C*-algebra $\mathcal{A}$ are quasi-equivalent if and only if their respective W*-representations $\pi^{w}_{\tilde{\varphi}}$ and $\pi^{w}_{\tilde{\psi}}$ are weakly (physically) equivalent.[79]

Quasi-equivalence requires in addition that the kernels of the W*-representations $\pi^{w}_{\tilde{\varphi}}$ and $\pi^{w}_{\tilde{\psi}}$ are equal. These results for C*-representations and W*-representations are summarized below.

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[79] Part of Emch's theorem establishes that $\pi_{\varphi}$ and $\pi_{\psi}$ are quasi-equivalent if and only if $\tilde{\pi}_{\varphi}$ and $\tilde{\pi}_{\psi}$ are physically equivalent. In Emch's notation (1972, 122), $\tilde{\pi}_{\varphi}$ is the unique ultraweakly continuous extension of $\pi_{\varphi}$ to the universal enveloping von Neumann algebra $\pi_{u}(\mathcal{A})''$. Emch also proves that $\tilde{\pi}_{\varphi}(\pi_{u}(\mathcal{A})'') = \pi_{\varphi}(\mathcal{A})''$ and by the discussion in section 4.1 $\pi_{\varphi}(\mathcal{A})'' = \pi^{w}_{\tilde{\varphi}}(\mathcal{A}^{**})$.

| C*-Representations: $\pi_{\omega}$ | | Associated W*-Representations: $\pi^{w}_{\tilde{\omega}}$ |
|---|---|---|
| Unitary Equivalence | $\Rightarrow$ | Unitary Equivalence |
| $\Downarrow$ | | $\Downarrow$ |
| Quasi-Equivalence | $\Leftrightarrow$ | Weak (Physical) Equivalence |
| $\Downarrow$ | | |
| Weak (Physical) Equivalence | | |

Table 4.4: Equivalence Relationships for C* and W*-Representations

Thus, the mere *weak* equivalence of two C*-representations is not enough to guarantee the *weak* equivalence of their respective W*-representations. What is the difference between W*-representations that are weakly (physically) *inequivalent*?

By theorem 4.6.2, $\pi^{w}_{\tilde{\varphi}}$ and $\pi^{w}_{\tilde{\psi}}$ are weakly *inequivalent* if and only if $\ker \pi^{w}_{\tilde{\varphi}} \neq \ker \pi^{w}_{\tilde{\psi}}$. If $\ker \pi^{w}_{\tilde{\varphi}} \neq \ker \pi^{w}_{\tilde{\psi}}$, then there are three possible cases: (i) $\pi^{w}_{\tilde{\varphi}}$ is faithful and $\pi^{w}_{\tilde{\psi}}$ is not faithful, (ii) $\pi^{w}_{\tilde{\varphi}}$ is not faithful and $\pi^{w}_{\tilde{\psi}}$ is faithful, and (iii) both $\pi^{w}_{\tilde{\varphi}}$ and $\pi^{w}_{\tilde{\psi}}$ are not faithful.[80] In any of these cases, there is an $A \in \mathcal{A}^{**}$ such that $A \neq 0$ and, for example in (ii), $\pi^{w}_{\tilde{\varphi}}(A) = 0$. It follows that for any density matrix $\rho_{\pi^{w}_{\tilde{\varphi}}}$ in the folium of $\pi^{w}_{\tilde{\varphi}}(\mathcal{A}^{**})$ there does not exist a density matrix $\rho_{\pi^{w}_{\tilde{\psi}}}$ in the folium of $\pi^{w}_{\tilde{\psi}}(\mathcal{A}^{**})$ that can approximate $\rho_{\pi^{w}_{\tilde{\varphi}}}$ for $A$, i.e. $\left|Tr\left(\rho_{\pi^{w}_{\tilde{\varphi}}} A\right) - Tr\left(\rho_{\pi^{w}_{\tilde{\psi}}} A\right)\right| \geq \varepsilon$.

Thus, there will always be some observable $A$ in the bidual $\mathcal{A}^{**}$ that will physically distinguish between any pair of weakly inequivalent W*-representations. It will be proved later in this chapter that this observable has classical properties.

Weakly inequivalent W*-representations must have quasi-equivalent C*-representation. The position of algebraic imperialism and its defense of physical equivalence nevertheless could be partially rescued if two unitarily inequivalent C*-representations are quasi-equivalent. The next section will show that this requirement is not satisfied by most physically interesting W*-representations.

### 4.7 TYPES OF VON NEUMANN ALGEBRAS

There are many different von Neumann algebras that are used in both AQFT and AQSM. This chapter will explore some of these important representations and how their properties make it impossible to be an algebraic imperialist in the face of UIRs.[81] It will be shown that for irreducible representations, KMS states, and type III factors that mere unitary inequivalence is sufficient to show that their W*-representations are not weakly equivalent.

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[80] Notice that for von Neumann algebras the faithfulness of a representation is no longer with respect to the original C*-algebra $\mathcal{A}$, but rather to the W*-algebra $\mathcal{A}^{**}$.

[81] That is if one is an algebraic imperialist with respect to C*-algebras. A kind of algebraic imperialism with respect to the bidual $\mathcal{A}^{**}$ will be discussed in the next chapter.

### 4.7.1 Irreducible Representations

Before turning to W*-representations, there are two important concepts necessary to understand representations: irreducibility and factors. Irreducibility can intuitively be thought of as cases where the representation is as "small" as possible. Some physicists view them as the most important kind of representation since they can serve as the building blocks for other representations. Whether one is looking for irreducible representations of the Lorentz group, the Schrödinger representation, or the Weyl representation, irreducibility has been a desideratum.[82] The advantages of using reducible representations will be examined in section 4.10. There are four equivalent ways to define irreducibility.

**Theorem 4.7.1.1:** A representation $\pi_\omega(\mathcal{A})$ of a C*-algebra $\mathcal{A}$ is called *irreducible* if any of the following conditions is satisfied.

(1) The only closed subspaces of $\mathcal{H}_{\pi_\omega}$ that are invariant under the action of the elements of $\pi_\omega(\mathcal{A})$ are $\mathcal{H}_{\pi_\omega}$ and **0**.
(2) The commutant $\pi_\omega(\mathcal{A})'$ consists of multiples of the identity operator, i.e., it is trivial: $\pi_\omega(\mathcal{A})' = \{\lambda \mathrm{I}\}$. (This is also called Schur's lemma.)
(3) $\omega$ is a pure state.
(4) Every non-zero vector in $\mathcal{H}_{\pi_\omega}$ is cyclic.

[82] According to Haag (1996, 54), one of the advantages of irreducible representations of the ETCCRs for quantum fields is that all observables can be expressed in terms of the quantum field and its conjugate momentum

The first condition implies that irreducible representations contain no nontrivial subrepresentations; they are the smallest possible representation. The commutant of irreducible representations is trivial and that implies that it has no nontrivial classical observables. The import of condition two is that the representation only contains “quantum” observables which only commute with the identity operator. Condition three implies that irreducible representations cannot be built using convex combinations of abstract states, i.e. $\omega \neq \lambda\varphi + (1-\lambda)\psi$ if $\pi_\omega(\mathcal{A})$ is irreducible. The last condition implies that any vector state in $\mathcal{H}_{\pi_\omega}$ is capable of being used to build the entire Hilbert space by applying the operators in $\pi_\omega(\mathcal{A})$ to it.

### 4.7.2 Disjoint Representations

The condition of disjointness is often used in discussing unitary inequivalence. It is usually contrasted with quasi-equivalence, but quasi-equivalence and disjointness are not mutually exclusive concepts.

**Theorem 4.7.2.1:** Let $\mathcal{A}$ be a C*-algebra and $\pi_\varphi$ and $\pi_\psi$ be two representations of $\mathcal{A}$. If any of the following conditions is satisfied, then $\pi_\varphi$ and $\pi_\psi$ are disjoint.

(1) No subrepresentation of $\pi_\varphi$ is unitarily equivalent to a subrepresentation of $\pi_\psi$.

(2) $C_{\tilde{\varphi}}C_{\tilde{\psi}} = 0 = C_{\tilde{\psi}}C_{\tilde{\varphi}}$

(3) The intersection of their folia is empty: $\mathfrak{F}_{\varphi} \cap \mathfrak{F}_{\psi} = \varnothing$.

Condition two says that disjoint representations have orthogonal central projections. The third condition implies that no density operator in $\mathcal{H}_{\pi_{\varphi}}$ can be expressed as a density operator in $\mathcal{H}_{\pi_{\psi}}$ and vice versa. This condition has been used by some (Arageorgis, Earman, and Ruetsche 2002b) to argue that UIRs are incommensurable physical theories and that the set of worlds physically possible for one representation are not physically possible worlds for a disjoint representation (Ruetsche 2003) and vice versa. A critical analysis of this argument will be given in the next chapter.

To illustrate that there are intermediate cases between quasi-equivalence and disjointness let $\pi_{\varphi}$ and $\pi_{\psi}$ be non-trivial unitarily inequivalent irreducible representations and suppose that $\pi_{\omega} = \pi_{\varphi} \oplus \pi_{\psi}$. The only nontrivial subrepresentation of $\pi_{\varphi}$ is $\pi_{\varphi}$ since it is irreducible. $\pi_{\omega}$ has two nontrivial subrepresentations: $\pi_{\varphi}$ and $\pi_{\psi}$. Since $\pi_{\varphi}$ and $\pi_{\psi}$ are unitarily inequivalent, every subrepresentation of $\pi_{\omega}$ is not unitarily equivalent to each subrepresentation of $\pi_{\varphi}$. By condition (3) of theorem 4.6.1, $\pi_{\omega}$ and $\pi_{\varphi}$ are *not* quasi-equivalent. However, $\pi_{\varphi}$ is unitarily equivalent to one subrepresentation of $\pi_{\omega}$, namely itself $\pi_{\varphi}$! Since there is a subrepresentation of $\pi_{\omega}$ and $\pi_{\varphi}$ that is unitarily equivalent, $\pi_\omega$ and $\pi_\varphi$ cannot be disjoint by condition (1) of theorem 4.7.2.1. Thus, $\pi_\omega$ and $\pi_\varphi$ are neither quasi-equivalent nor disjoint.

### 4.7.3 Factor Representations

The basic classification of von Neumann algebras was carried out by Murray and von Neumann (1936).[83] The building blocks of von Neumann algebras are *factors*. To understand factors, the concept of the center must be introduced.

**Definition:** The *center* $\mathfrak{Z}_{\pi_\omega(\mathcal{A})''}$ of a von Neumann algebra $\pi_\omega(\mathcal{A})''$ is the intersection of the von Neumann algebra and its commutant $\pi_\omega(\mathcal{A})'$, i.e. $\mathfrak{Z}_{\pi_\omega(\mathcal{A})''} = \pi_\omega(\mathcal{A})'' \cap \pi_\omega(\mathcal{A})'$.

**Definition:** A *factor* is a von Neumann algebra with a trivial center, i.e. $\mathfrak{Z}_{\pi_\omega(\mathcal{A})''} = \pi_\omega(\mathcal{A})'' \cap \pi_\omega(\mathcal{A})' = \{\lambda \mathrm{I}_{\pi_\omega}\}$. In other words, a factor's center consists of scalar multiples of the Hilbert space $\mathcal{H}_{\pi_\omega}$ identity operator $\mathrm{I}_{\pi_\omega}$.

If a C*-representation is irreducible, then its von Neumann algebra is a factor. This follows because if $\pi_\varphi$ is irreducible, then by condition (2) of theorem 4.7.1.1

[83] Von Neumann and Murray were able to construct examples of the three basic types of factors (types I, II, III), but they were not able to completely subclassify the type III factor. Connes won the Fields Medal for his subclassification of the type III factors some fifty years after the Murray-von Neumann papers on "rings of operators."

$\pi_\varphi(\mathcal{A})' = \{\lambda \mathrm{I}_{\pi_\varphi}\}$. This implies that $\mathfrak{Z}_{\pi_\varphi} = \pi_\varphi(\mathcal{A})'' \cap \pi_\varphi(\mathcal{A})' = \{\lambda \mathrm{I}_{\pi_\varphi}\}$, so $\pi_\varphi(\mathcal{A})''$ is a factor by the definition above. Thus, irreducibility implies being a factor.

| |
|---|
| Irreducible Representation |
| $\Downarrow$ |
| Factor Representation |

Table 4.5: Relationship between Irreducibility and Factor Representations

If two representations are factors, then they are either quasi-equivalent or disjoint. Thus, being a factor makes quasi-equivalence and disjointness mutually exclusive concepts.

> **Theorem 4.7.3.1** (Proposition 10.3.12 (ii), Kadison and Ringrose 1997b)**:** Let $\mathcal{A}$ be a C*-algebra and $\pi_\varphi$ and $\pi_\psi$ be two factor representations of $\mathcal{A}$. $\pi_\varphi$ and $\pi_\psi$ are either quasi-equivalent or disjoint.

There are three factors types: type **I**, **II**, **III**. The rough idea behind the classification is the following.[84] Every von Neumann algebra is generated from its projections. A dimension function can be defined to classify the range of the projections. Murray and von Neumann proved that the dimension function *d* of the projections can have the following ranges.

[84] For more technical details, see chapter 6 of Kadison and Ringrose (1997b).

**Type $\mathbf{I_n}$:** *d*: {0,1,…,n}, where n is a natural number
*Physical Example*: non-relativistic finite dimensional quantum mechanics (n x n complex matrices)

**Type $\mathbf{I_\infty}$:** *d*: {0,1,…,$\infty$}
*Physical Example*: Free boson Fock space

**Type $\mathbf{II_1}$:** *d*: [0,1]
*Physical Examples*: Free fermion Fock space
Infinite temperature maximally chaotic KMS state

**Type $\mathbf{II_\infty}$:** *d*: [0,$\infty$)
*Physical Example*: The tensor product of the free boson and the free fermion Fock spaces

**Type III:** *d*: {0,$\infty$} (two-element set)
*Physical Examples*: Local algebras in AQFT
KMS states with finite non-zero temperature in AQSM

Table 4.6: Factor Types and Examples

Every von Neumann algebra can be written as a direct sum, or direct integral, of types I, II, and/or III factors. While the type III factor seems to be pathological, "[t]o the malicious delight of mathematicians" (Thirring 1983, 41) type III factors are the most useful factor for modeling many physical systems. In particular, local algebras, which are defined on open regions of Minkowski spacetime, are generally of type III. A KMS state $\phi$ that corresponds to a pure phase of an infinite system at a finite temperature $\beta$ induces a representation $\pi_\phi$ that is also a type III factor.[85]

It should now be clear that W*-representations, and in particular type III factors, are the most important representation. C*-representations are not sufficient to model many of the important physical examples in AQFT and AQSM. Thus, the algebraic imperialist claims about physical equivalence and C*-representations cannot be used by themselves as arguments that all of the physical content of QFT resides in the abstract algebra. Indeed, many philosophers of physics have criticized the notion of physical equivalence on the grounds that there are representation dependent features such as symmetry breaking (Arageorgis 1995, 159) and phase transitions (Ruetsche 2003, 1138) that are part of the physical content of a theory. But their arguments do not strike a critical blow against the algebraic imperialist since they do not show the mathematical and physical limitations of Haag and Kastler's notion of physical equivalence much less the consequences of this breakdown. The physical equivalence of two C*-representations does not guarantee that their W*-representations will be physically equivalent. If the C*-representations are not quasi-equivalent, then their W*-representations will not be physically equivalent. The algebraic imperialist might hope that quasi-equivalence could be used to banish UIRs to the realm of mathematical annoyances. Unfortunately for the algebraic imperialist, the distinction between unitary equivalence and quasi-

85 The KMS condition is discussed in much more detail in section 4.7.4.

equivalence collapses for many physically useful representations. *In these cases, mere unitary inequivalence is enough to guarantee the weak (physical) inequivalence of the W*-representations.*

For example, irreducibility collapses the distinction between unitary equivalence and quasi-equivalence and also the distinction between disjointness and unitary inequivalence. This was proved by (Kadison and Ringrose 1997b, 740).[86]

> **Theorem 4.7.3.2:** Suppose that $\pi_\varphi$ and $\pi_\psi$ are irreducible representations of a C*-algebra $\mathcal{A}$.
> (i) $\pi_\varphi$ and $\pi_\psi$ are unitarily equivalent if and only if they are quasi-equivalent.
> (ii) $\pi_\varphi$ and $\pi_\psi$ are unitarily inequivalent if and only if they are disjoint.

This implies that if two C*-representations are unitarily *inequivalent*, then they are not quasi-equivalent, and, by theorem 4.6.2 their W*-representations are not physically equivalent. This leads to the following generalization.

> **Theorem 4.7.3.3**: Suppose that $\pi_\varphi$ is either irreducible or a factor representation of a C*-algebra $\mathcal{A}$ and $\pi_\psi$ is either irreducible or a factor representation of a C*-algebra $\mathcal{A}$. $\pi_\varphi$ and $\pi_\psi$ are unitarily *inequivalent* if and only if $\pi^w_{\tilde{\varphi}}$ and $\pi^w_{\tilde{\psi}}$ are *not* physically equivalent.

[86] Kadison and Ringrose use the term "equivalent" instead of "unitarily equivalent" in their statement of the theorem, but their use of "equivalent" means the same thing as "unitarily equivalent." I prefer the more explicit wording.

Proof: Irreducibility implies being a factor representation, so Theorem 4.7.3.1 implies that as long as $\pi_\varphi$ and $\pi_\psi$ are irreducible or factors, then they are disjoint if and only if they are not quasi-equivalent. Since it is assumed that they are disjoint, they are not quasi-equivalent. Thus, by Theorem 4.6.2, this implies that $\pi^w_{\tilde{\varphi}}$ and $\pi^w_{\tilde{\psi}}$ cannot be physically equivalent.

Irreducible representations are not the only types of representations for which the distinction between unitary and quasi-equivalence collapses. It also collapses for any representation of a KMS state and for any type III factor representation as long as the type III factor represents the observables as bounded operators on a *separable* Hilbert space.

### 4.7.4 KMS Representations

Equilibrium states play a crucial role in statistical mechanics and especially in AQSM. Traditionally, equilibrium states are associated with the Gibbs state of a system enclosed in a finite box with volume *V*. The three most common ways to define an equilibrium state are the microcanonical ensemble, canonical ensemble, and the grand canonical ensemble.[87] Each can be

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[87] My discussion of KMS states is drawn primarily from (Bratteli and Robinson 1997), (Emch 1984) and (Primas 1983).

described in AQSM, but the Gibbs grand canonical equilibrium state $\rho_V$ is more convenient. It is defined as

$$\rho_V = \frac{e^{-\beta\left(H_V - \sum_j \mu_j N_{jV}\right)}}{Tr\left[e^{-\beta\left(H_V - \sum_j \mu_j N_{jV}\right)}\right]}$$

where $H_V$ is the Hamiltonian of the system enclosed in a box of volume *V*, $N_{jV}$ is the number operator for the *j*-th species of particles in the box, $\beta = \dfrac{1}{kT}$ is the inverse temperature *T* (*k* is Boltzman's constant), and $\mu_j$ is the chemical potential for the *j*-th species of particles. For finite systems, the Gibbs state is well-defined and unique (Ruelle 1969). Thus, there is a kind of Stone-von Neumann theorem at work for finite systems in equilibrium in that the Gibbs state is unique. However, this uniqueness comes at a price since large but finite systems cannot accommodate phase transitions, which require multiple distinct equilibrium states such as liquids and vapor, nor can it model ergodic behavior. The traditional way to add these additional states is to take the thermodynamic limit where the number of particles $N \to \infty$ and the volume $V \to \infty$ but their ratio $0 < \dfrac{N}{V} < \infty$.

The Stone-von Neumann theorem does not hold in the thermodynamic limit since the number of degrees of freedom becomes infinite. The non-uniqueness of equilibrium states allows for multiple states to be used to model phase transitions and ergodic behavior.

The problem with using Gibbs states as equilibrium states is that they are often not well-defined. For example, the Gibbs state does not exist if the spectrum of $H_V$ fails to be pure discrete (see section 10.1a of (Emch 1984)). In general, there is no Hamiltonian *H* or number operator *N* that is a member of the C*-algebra $\mathcal{A}$ such that the time evolution $\alpha_t$ of an operator *A* can be generated from them via $\alpha_t(A) = e^{i\left(H - \sum_j \mu_j N_j\right)t} A e^{-i\left(H - \sum_j \mu_j N_j\right)t}$. This type of dynamics only holds for very simple models such as a noninteracting Fermi gas. In general, it will not hold for infinite systems where initial states can accumulate an infinite number of particles and energy locally (Bratteli and Robinson 1997, 4). Also, the Gibbs state does not usually exist in the thermodynamic limit, i.e. $\lim_{V \to \infty} \rho_V = 0$ (Hugenholtz 1973, 290). To deal with these issues, a more general notion of equilibrium states had to be found.

In AQSM, the expectation value of the Gibbs grand canonical equilibrium state $\omega_{\beta,\mu}$ is defined as:

$$\omega_{\beta,\mu_j}(A) = Tr(\rho_V A) = \frac{Tr\left(e^{-\beta\left(H_V - \sum_j \mu_j N_{jV}\right)} A\right)}{Tr\left(e^{-\beta\left(H_V - \sum_j \mu_j N_{jV}\right)}\right)}.$$

If the Gibbs state is well-defined and it is assumed that $\rho_V$ is a trace-class operator, *H* is lower semi-bounded, and the trace-class property is valid for all

$\beta > 0$, then using the invariance of a trace for cyclic permutations it can be shown that this state satisfies the Kubo-Martin-Schwinger, or KMS condition defined below.

Given the problems of having a well-defined Gibbs state describe equilibrium states, states that satisfy the KMS condition are taken to be equilibrium states for reasons to be explained shortly. A KMS state requires a real one-parameter group of automorphisms $\alpha_t$ of the C*-algebra $\mathcal{A}$ that encodes the dynamics through time *t*. An abstract state $\omega_{\beta,\mu}$ is a KMS state with respect to an automorphism group $\alpha_t$, inverse temperature $\beta$, and chemical potential $\mu_j$ for each particle species *j* if and only if the following holds for all operators A, B in a dense subalgebra of $\mathcal{A}$.

$$\omega_{\beta,\mu_j}\left[\mathrm{A}\alpha_{i\beta}(\mathrm{B})\right] = \omega_{\beta,\mu_j}(\mathrm{BA})$$

This equation is the KMS condition and it uniquely determines with respect to $\alpha_t$ with value $\beta$ the Gibbs grand canonical equilibrium state. These states will be referred to as $(\alpha_t, \beta, \mu_j)$-KMS states.

There are several reasons why $(\alpha_t, \beta, \mu_j)$-KMS states are associated with equilibrium states. (1) For finite volume systems, the Gibbs states satisfy the KMS condition. (2) If ($\mathcal{A}$, $\alpha_t$) admits a Gibbs state at inverse temperature $\beta$, then the $(\alpha_t, \beta, \mu_j)$-KMS state is unique and coincides with the Gibbs state (Bratteli and Robinson 1997, 119). (3) The KMS condition is stable under limits.

In a fundamental AQSM paper (Haag, Hugenholtz, and Winnink 1967), it was proved under certain assumptions that the limiting state obtained by taking the thermodynamic limit satisfies the KMS condition. In fact, for lattice systems the equivalence of the Gibbs and KMS condition was established (Araki and Ions 1974). (4) States in equilibrium should not change through time, in other words they should be *stationary*. Every KMS state is time-translation invariant, i.e., $\rho\left(\alpha_t\left(A\right)\right)=\rho\left(A\right)$ for all $A \in \mathcal{A}$ (Winnink 1968). (5) KMS states are stable under all local perturbations of the dynamics by a perturbing Hamiltonian (Araki 1973). For infinite systems, given that a state is stationary, stable under local perturbations, and relatively pure, it was proved that the state must satisfy the KMS condition (Haag, Kastler, and Trych-Pohlmeyer 1974). (6) Passive states are defined as states from which energy cannot be extracted in any cyclic process using local perturbations in a finite interval of time. It can be proven that a factor state satisfies the KMS condition if and only if it is passive (see discussion (Primas 1983, 185)). For all these reasons the KMS condition is taken as an "empirical definition" of an equilibrium state (Bratteli and Robinson 1997, 5).

KMS states are crucial for AQSM, but they are useful in AQFT as well. Later in the chapter, their connection with the Unruh effect will be explained. In what follows, it will be shown that KMS states are crucial to providing counterexamples to physical equivalence. Of more immediate importance is the following proof which shows that the distinction between unitary equivalence and quasi-equivalence collapses for KMS states.

> **Theorem 4.7.7:** Let $\varphi$, $\psi$ be KMS states. $\pi_\varphi$ and $\pi_\psi$ are quasi-equivalent if and only if they are unitarily equivalent.

### 4.7.5 Type III Factors

As discussed before, type III factors are commonplace in both AQFT and AQSM. The next theorem shows that the distinction between unitary equivalence and quasi-equivalence collapses for type III factors given that the Hilbert space constructed via the GNS theorem is separable. This is not an unreasonable restriction since most models used in AQFT and AQSM assume the Hilbert space is separable.[88]

> **Theorem 4.7.8:** Let $\pi_\varphi$ and $\pi_\psi$ be type III factor representations of a C*-algebra $\mathcal{A}$ into the bounded operators $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is separable. $\pi_\varphi$ and $\pi_\psi$ are quasi-equivalent if and only if they are unitarily equivalent.

[88] This is *not* to say that non-separable Hilbert spaces have no use in QFT. In the previous chapter, the tensor product of the continuum of possible Fock spaces is non-separable. This "universal receptacle" will be discussed more in the next chapter.

Thus, for irreducible, KMS, or type III factor representations mere unitary *inequivalence* is sufficient to show that their W*-representations are not physically equivalent.[89]

| C*-Representations: $\pi_\omega(\mathcal{A})$ | | Associated W*-Representations: $\pi^w_{\tilde{\omega}}$ |
|---|---|---|
| Unitary Equivalence<br>$\Downarrow$<br>Weak (Physical) Equivalence | $\Leftrightarrow$ | Weak (Physical) Equivalence |

Table 4.7: Equivalence Relationships for Irreducible, KMS states, and Type III Factors

These results imply the following.

**Theorem 4.7.9**: Assume $\pi_\varphi$ and $\pi_\psi$ are irreducible representations, or type III factor representations on a separable Hilbert space $\mathcal{H}$, or KMS representations of a C*-algebra $\mathcal{A}$. They are unitarily inequivalent if and only if $\pi^w_{\tilde{\varphi}}$ and $\pi^w_{\tilde{\psi}}$ are not physically equivalent.

Proof: By theorems 4.7.3.2 (i), 4.7.7, and 4.7.8, if both representations are unitarily inequivalent, then they are not quasi-equivalent. Since they are not quasi-equivalent, then by theorem 4.6.2, their W*-representations are not physically equivalent.

This theorem shows that mere unitary inequivalence is enough to generate physically inequivalent representations for some of the most useful

[89] A KMS representation is the representation generated from an abstract state which satisfies the KMS condition.

representations: irreducible representations, KMS representations, and type III factors representations.

### 4.7.6 Counterexamples: Temperature and Chemical Potential

Up to this point the focus has been on undermining Haag and Kastler's notion of physical equivalence at a general level. This section will look at specific physical examples of UIRs that do not satisfy the condition of physical equivalence for von Neumann algebras. The first result was proved by Takesaki (1970).

**Theorem 4.7.6.1:** Let $\varphi$ and $\psi$ be $\left(\alpha_t, \beta, \mu_j\right)$ and $\left(\alpha_t, \gamma, \mu_j\right)$- KMS states with inverse temperatures $\beta$ and $\gamma$ respectively such that $\beta \neq \gamma$. Assume that either $\pi_\varphi$ or $\pi_\psi$ is a type III factor representation. Then $\pi_\varphi$ and $\pi_\psi$ are disjoint.

**Corollary 4.7.6.2:** Let $\varphi$ and $\psi$ be KMS states with inverse temperatures $\beta$ and $\gamma$ respectively such that $\beta \neq \gamma$. Assume that either $\pi_\varphi$ or $\pi_\psi$ is a type III factor representation. Then $\pi_{\tilde{\varphi}}^w$ and $\pi_{\tilde{\psi}}^w$ are not physically equivalent.

Proof: Since $\pi_\varphi$ and $\pi_\psi$ are disjoint by theorem 4.7.6.1, they are not quasi-equivalent by theorem 4.7.3.1. Thus, by theorem 4.6.2, $\pi_{\tilde{\varphi}}^w$ and $\pi_{\tilde{\psi}}^w$ are *not* physically equivalent.

The assumption that one of the factors is type III is not a severe restriction since the von Neumann algebra generated by a KMS state with a nonzero finite temperature will be a type III factor (Hugenholtz 1967). This only leaves two cases: (i) the temperature is zero, in which case the representation is type I, and (ii) the temperature is infinite, in which case the representation is type $\mathrm{II}_1$.[90] The representations in both cases will be disjoint from any representation of a KMS state with a nonzero finite temperature. (i) and (ii) will also be disjoint from each other since they are different factor types.

Corollary 4.7.6.2 is a very rich source of UIRs which are not physically equivalent von Neumann algebras. Given any two KMS states with different nonzero finite temperatures, which implies that the associated representations are type III factors, they will be unitarily inequivalent and their W*-representations will not be physically equivalent no matter how small the difference is between their temperatures. Thus, there will be a continuum of UIRs whose von Neumann algebras will not be physically equivalent!

Another source of UIRs in AQSM comes from a theorem of Müller-Herold (1980) for chemical potential.

> **Theorem 4.7.6.3:** Let $\varphi$ and $\psi$ be $(\alpha_t, \beta, \mu)$ and $(\alpha_t, \beta, \tau)$-KMS states respectively and assume $\mu \neq \tau$.[91] Then $\pi_\varphi$ and $\pi_\psi$ are disjoint.[92]

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[90] KMS states with negative temperatures are mathematically possible as well as *ceiling states* (Bratteli and Robinson 1997, 97), which have inverse temperatures $\beta = -\infty$. These will be not be considered in this dissertation.

[91] Müller-Herold assumed that there is only one type of particle species, so the chemical potential in section 4.6.4 for multiple particle species $\mu_j$ becomes just $\mu$.

**Corollary 4.7.6.4:** Let $\varphi$ and $\psi$ be $(\alpha_t,\beta,\mu)$ and $(\alpha_t,\beta,\tau)$-KMS states respectively and assume $\mu \neq \tau$. Then $\pi^w_{\tilde{\varphi}}$ and $\pi^w_{\tilde{\psi}}$ are not physically equivalent.

Proof: Since $\pi_\varphi$ and $\pi_\psi$ are disjoint by theorem 4.7.6.3, they are not quasi-equivalent by theorem 4.7.3.1. Thus, by theorem 4.6.2, $\pi^w_{\tilde{\varphi}}$ and $\pi^w_{\tilde{\psi}}$ are *not* physically equivalent.

As in the case of temperature, this theorem provides a continuum of UIRs whose W*-representations are physically inequivalent no matter how small the difference between their chemical potentials is. This shows that KMS states are extremely sensitive to variations in temperature or chemical potential. Though these examples show how to generate UIRs whose W*-representations are physically inequivalent in AQSM, there are similar examples in AQFT. Later in this chapter it will be shown that there is a continuum of physically inequivalent W*-representations for the Unruh effect.

### 4.7.7 The Intuitive Failure of Physical Equivalence

It has now been rigorously proven that there are numerous counterexamples to Haag and Kastler's claim that their notion of physical equivalence shows that UIRs are physically insignificant. Now it will be shown

[92] Müller-Herold's result, unlike Takesaki's theorem, does not assume that one of the

that Haag and Kastler's notion of physical equivalence (HK-physical equivalence) does not capture our intuitive concept of when two systems are physically equivalent (IC-Physical Equivalence).[93] Consider two systems alike in every respect except that the difference in their temperatures is as small you want. Make the difference so small that measuring devices cannot measure it. Both systems are IC-physically equivalent. However, corollary 4.7.6.2 shows that these two systems would be HK-physically *inequivalent*. Another failure of HK-physical equivalence can be given by considering Haag's theorem.[94] One way of stating Haag's theorem is that the free and interacting representations are unitarily inequivalent. But if HK-physical equivalence is applicable to all UIRs, then a free representation and an interacting representation would be classified as physically equivalent. However, using our IC-physical equivalence it would be crazy to suggest that a system where no interactions occur is physically equivalent to a system where there are interactions. Thus, HK-physical equivalence does not apply to temperature or chemical potential and it fails to coincide with our IC-physical equivalence.

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representations is type III.

[93] Sometimes two systems are called physical equivalence when an isomorphism exists between their states and observables (see (Ruetsche 2006)). This metaphysical characterization of physical equivalence is different from the intuitive notion of physical equivalence used in this section, which is based on the intuitive concept of physical equivalence used by pragmatically-minded physicists. For these physicists, isomorphism is too strong a requirement for physical equivalence. Rather, two systems are physically equivalent if they are roughly the same kind of system with basically the same features.

[94] I would like to thank David Baker for a helpful conversation on this topic.

### 4.7.8 Explanation of the Failure of HK-Physical Equivalence

As discussed at the end of section 4.6, the physical inequivalence of two W*-representations implies that there is an element on which the two representations will differ with respect to the expectation values of their states. They are operationally different. Anticipating the results in the next section, it will be proved that this element belongs to the center of the bidual and that it has two properties associated with classical observables. It will turn out that HK-physical equivalence does have some usefulness, namely as a criterion for the *classical equivalence* of two W*-representations.

## 4.8 CLASSICAL EQUIVALENCE AND PHYSICAL EQUIVALENCE

The argument that HK-physical equivalence shows that there are no significant physical differences between different UIRs has now been demolished. At the theoretical level, the physical equivalence of W*-representations requires that the C*-representations be quasi-equivalent, which in many cases reduces to the C*-representations being unitarily equivalent. More specifically, it has been shown that a continuum of UIRs can be constructed for which any two W*-representations will be physically inequivalent to each other. Does HK-physical equivalence have any usefulness? The answer is yes. To borrow a phrase from Ruetsche (2003), HK-physical equivalence can be “put to work.” In fact, it shows us where to locate the physical differences between any two UIRs. A general theorem is proven below which shows that two W*-representations which are factors are physically equivalent if and only if they assign the same values to all classical – not quantum – properties. The proof of this assertion requires a preliminary lemma and a theorem.

> **Lemma 4.8.1:** Let $\varphi$ be a state on the C*-algebra $\mathcal{A}$, $\tilde{\varphi}$ the extension of $\varphi$ to be a normal state on the W*-algebra $\mathcal{A}^{**}$, and denote $\mathfrak{Z}(\mathcal{A}^{**})$ as the center of $\mathcal{A}^{**}$. Then each condition implies the condition below it.[95]
>
> (1) $\varphi$ is primary[96]
> (2) $\pi_{\tilde{\varphi}}(Z) = \tilde{\varphi}(Z)\mathrm{I}_{\pi^w_{\tilde{\varphi}}}$ where $\mathrm{I}_{\pi^w_{\tilde{\varphi}}}$ is the identity operator in $\mathcal{H}_{\pi^w_{\tilde{\varphi}}}$ and $Z \in \mathfrak{Z}(\mathcal{A}^{**})$
> (3) $\tilde{\varphi}(AZ) = \tilde{\varphi}(A)\tilde{\varphi}(Z) = \varphi(A)\tilde{\varphi}(Z)$ for all $A \in \mathcal{A}$ and $Z \in \mathfrak{Z}(\mathcal{A}^{**})$
> (4) the restriction of $\tilde{\varphi}$ to $\mathfrak{Z}(\mathcal{A}^{**})$ is pure

Using this lemma, the following theorem can be proved.

> **Theorem 4.8.2:** Let $\varphi$, $\psi$ be two primary states. $\pi_\varphi$ and $\pi_\psi$ are quasi-equivalent if and only if for every observable $Z \in \mathfrak{Z}(\mathcal{A}^{**})$, $\tilde{\varphi}(Z) = \tilde{\psi}(Z)$.

The assumption that $\pi_\varphi$ and $\pi_\psi$ are factors is only necessary for the ($\Rightarrow$) part of the proof. The converse of this theorem, namely that $\varphi$, $\psi$ are disjoint primary states if and only if there exists an observable $Z \in \mathfrak{Z}(\mathcal{A}^{**})$, $\tilde{\varphi}(Z) \neq \tilde{\psi}(Z)$, shows

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95 These conditions should be equivalent to each other, but I have not been able to find a satisfactory proof that condition (4) implies condition (1).
96 To say that an abstract state $\varphi$ is primary is equivalent to saying that the von Neumann algebra $\pi_\varphi(\mathcal{A})''$ it generates is a factor.

that there exists an element that will distinguish different UIRs. The following corollary can be proven by applying theorem 4.8.2 to HK-physical equivalence.

> **Corollary 4.8.3:** Let $\varphi$, $\psi$ be two primary states. The W*-representations, $\pi_{\tilde{\varphi}}^{w}$ and $\pi_{\tilde{\psi}}^{w}$, are HK-physically equivalent if and only if for every observable $Z \in \mathfrak{Z}\left(\mathcal{A}^{**}\right)$, $\tilde{\varphi}(Z) = \tilde{\psi}(Z)$.

The converse for this corollary shows that two HK-physically *inequivalent* W*-representations will be distinguished from each other by some observable in $\mathfrak{Z}\left(\mathcal{A}^{**}\right)$. The physical equivalence of two W*-representations is sufficient for their being *classically equivalent* to each other since their states will have *exactly* the same expectation values for every observable in $\mathfrak{Z}\left(\mathcal{A}^{**}\right)$. Thus, HK-physical equivalence for W*-representations does have some value as a criterion for when two W*-representations are classically equivalent.

HK-physical equivalence is both too weak and too strong to accomplish the aims Haag and Kastler have for it. It is too weak in that the mere HK-physical equivalence of two C*-representations is not sufficient to guarantee that the W*-representations they generate are HK-physically equivalent. As shown above, for many physically significant representations *mere unitary inequivalence* of two C*-representations is sufficient for their W*-representations to be HK-physically *inequivalent*. But the condition of HK-physical equivalence for W*-representations is too strong because it does not capture IC-physical equivalence. If two systems had *roughly* the same expectation values for *most* classical observables, then those systems would satisfy IC-physical equivalence. However, in order for two W*-representations to be HK-physically equivalent they must have the *exact* same expectation values for *every* classical observable according to corollary 4.8.3. But does the physical *inequivalence* of two W*-representations imply that they are classically *inequivalent* to each other? To put the matter another way, it is certainly a necessary condition for something to be a classical observable that it belongs to $\mathfrak{Z}(\mathcal{A}^{**})$. But is belonging to $\mathfrak{Z}(\mathcal{A}^{**})$ a sufficient condition for being a classical observable? Maybe the classical observables are only a subset of $\mathfrak{Z}(\mathcal{A}^{**})$. This is the topic of the next section.

### 4.9 WHERE ARE THE CLASSICAL OBSERVABLES?

Primas (2000, 166), Pöttinger (1989, 361), and Müller-Herold (1980, 45) claim that the classical observables are nontrivial, self-adjoint elements of the center of any von Neumann algebra. All of the nontrivial self-adjoint observables in the center of $\pi_u(\mathcal{A})''$ are classical observables for Primas. Unfortunately, none of these authors offers any argument why *every* observable belonging to the center of a von Neumann algebra $\pi_\omega(\mathcal{A})''$ is a classical observable. Presumably, the argument for them being classical observables is that they commute with every observable in $\pi_\omega(\mathcal{A})''$. While this is certainly a necessary condition for an observable to be classified as classical, it may not be a sufficient condition. It may be that the classical observables are a subset of the center of $\pi_u(\mathcal{A})''$ or belong to the center of a particular $\pi_\omega(\mathcal{A})''$. For example, Hepp claimed (1972, 241) that the classical observables correspond to operations which can be made outside of any bounded region of spacetime and thus identified the classical observables as belonging to the algebra of observables at infinity $\mathfrak{B}_{\pi_\varphi}$ relative to the representation $\pi_\varphi$, which is a subset of the center of $\pi_\varphi(\mathcal{A})''$. There are four different types of observables that belong to the center of a von Neumann algebra: (1) all of the center observables, (2) the observables at infinity, (3) macroscopic observables, and (4) globally macroscopic observables. This section will examine each type. All of the nontrivial elements in the center of $\pi_u(\mathcal{A})''$ will have abstract counterparts in $\mathfrak{Z}(\mathcal{A}^{**})$. Using corollary 4.8.3, HK-physical equivalence can be put to work as a condition for two representations being *classically* or *macroscopically equivalent*. In the next section, I will show how to make belonging to the center a sufficient condition for being a classical observable.

#### 4.9.1 Center Observables

If classical observables are non-trivial elements of the center of a von Neumann algebra, then the von Neumann algebra cannot be a factor. If it is not a factor, then the representation is not irreducible. Non-trivial center observables are elements of reducible representations. This means that the representations which contain non-trivial center observables are direct sums or direct integrals of representations. $\pi_u(\mathcal{A})$ is a direct sum of all representations, so $\pi_u(\mathcal{A})''$ has the largest collection of non-trivial center observables.

### 4.9.2 Observables at Infinity

One of the central ideas of AQFT is that the algebra of observables $\mathcal{A}$ is defined on open regions of $O$ Minkowski spacetime $\mathcal{M}$ with compact closure, which is denoted as $\mathcal{A}(O)$. The observables of $\mathcal{A}(O)$ are strictly local observables. The open regions are usually double-cones, which are nonempty intersections of the interiors $O$ of a forward and backward light cone. By taking the set theoretic union of all $\mathcal{A}(O)$ in $\mathcal{M}$ and closing in the norm topology the algebra of all quasi-local observables can be defined: $\mathcal{A}_{\text{loc}} = \overline{\bigcup_{O\in\mathcal{M}} \mathcal{A}(O)}$. Let $O^{\perp}$ be a region of $\mathcal{M}$ that is causally disjoint from $O$, i.e. $O \cap O^{\perp} = \varnothing$. A concrete C*-algebra $\pi_{\varphi}\left(\mathcal{A}\left(O^{\perp}\right)\right)$ and a von Neumann algebra $\pi_{\varphi}\left(\mathcal{A}\left(O^{\perp}\right)\right)''$ can be associated with $O^{\perp}$. Let $\mathfrak{B}_{\pi_{\varphi}}(O) = \pi_{\varphi}\left(\mathcal{A}\left(O^{\perp}\right)\right)''$. The *algebra of observables at infinity relative to* $\pi_{\varphi}$ is constructed by taking the intersection of all of these regions: $\mathfrak{B}^{\infty}_{\pi_{\varphi}} = \bigcap_{O\in\mathcal{M}} \mathfrak{B}_{\pi_{\varphi}}(O)$. Notice that this algebra is a von Neumann algebra that can be defined for any representation. This algebra can also be defined in AQSM by replacing Minkowksi spacetime $\mathcal{M}$ with Euclidean 3-space $\mathbb{R}^3$:

$\mathfrak{B}^{\infty}_{\pi_\varphi} = \bigcap_{O \in \mathbb{R}^3} \mathfrak{B}_{\pi_\varphi}(O)$. It can be proven (Emch 1984, 395) that the algebra of observables at infinity relative to $\pi_\varphi$ is a subset of its center: $\mathfrak{B}^{\infty}_{\pi_\varphi} \subseteq \mathfrak{Z}_{\pi_\varphi}$. $\mathfrak{B}^{\infty}_{\pi_\varphi}$ is in the center of the universal enveloping von Neumann algebra of the quasi-local C*-algebra of observables $\pi_u(\mathcal{A}_{\text{loc}})''$. However, the set of observables at infinity is often trivial (i.e., it contains only scalar multiples of the identity). Emch proved[97] (1984, 394-397) that every primary state $\varphi$ on $\mathcal{A}_{\text{loc}}$ is uniformly clustering[98] and that the property of uniform clustering implies that $\mathfrak{B}^{\infty}_{\pi_\varphi}$ is trivial. Since the center of $\pi_\varphi$ is trivial and $\mathfrak{B}^{\infty}_{\pi_\varphi}$ is trivial, this implies that $\mathfrak{B}^{\infty}_{\pi_\varphi} = \mathfrak{Z}_{\pi_\varphi}$.

### 4.9.3 Macroscopic Observables

Macroscopic observables are special observables at infinity according to Hepp (1972, 241). They are constructed as follows. For any sequence $O$ of bounded regions of $\mathcal{M}$ or $\mathbb{R}^3$ converging to infinity, such that all $O$ lie outside of any bounded region,[99] let $A_n \in \mathcal{A}(O_n)$ with $\|A_n\| \le b$ such that $A_n$ is a uniformly bounded sequence. If the weak limit $w - \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \pi_\varphi(A_n) = A$ exists, then

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97 Emch assumed local commutativity, but that condition played no part in his proof.
98 Hepp (1972, 241) referred to this condition as a state having short range correlations.
99 Another way to phrase this condition is that the distance of $O$ tends to infinity with $n$.

$A \in \mathfrak{B}_{\pi_\varphi}^{\infty}$. Thus, the set of macroscopic observables $\mathfrak{M}_{\pi_\varphi}$ is a subset of $\mathfrak{B}_{\pi_\varphi}^{\infty}$ which is itself a subset of $\mathfrak{Z}_{\pi_\varphi}$: $\mathfrak{M}_{\pi_\varphi} \subseteq \mathfrak{B}_{\pi_\varphi}^{\infty} \subseteq \mathfrak{Z}_{\pi_\varphi}$.

### 4.9.4 Global Macroscopic Observables

Sewell (2002, 42-43) constructed what he called *global macroscopic observables* $\mathcal{G}$ which are globally intensive observables of infinitely extended systems. They are not elements of $\mathcal{A}_{\text{loc}}$; they are functionals on state space. The simplest global macroscopic observables are infinite volume limits of observables $A_O$ given by space averages of local observables over bounded spatial regions $O$: $A_O = \frac{1}{|O|}\int_O A(x)\,dx$, where $|O|$ is defined as $\int_O dx$ or rather the volume of $O$.[100] The family of states $\varphi$ for which the limit as the bounded region $O$ approaches the entire space $\mathcal{M}$ or $\mathbb{R}^3$, $\lim_{O \to \mathcal{M} \text{ or } \mathbb{R}^3} \varphi(A_O)$, exists defines an observable $A(\varphi) = \lim_{O \to \mathcal{M} \text{ or } \mathbb{R}^3} \varphi(A_O)$ that is a functional on the state space of $\varphi$. $A(\varphi)$ is a global macroscopic observable since it corresponds to a global average of $A(x)$; it is the smearing out of the observable over the entire space. As Sewell showed (2002, 43), Hepp's definition of a macroscopic observable reduces to his definition of a globally macroscopic observable as a special case.

Thus, the set of global macroscopic observables for $\varphi$, denoted as $\mathcal{G}_{\pi_\varphi}$, is a subset of $\mathfrak{M}_{\pi_\varphi}$. For any von Neumann algebra, the following relations hold.

$$\mathcal{G}_{\pi_\varphi} \subseteq \mathfrak{M}_{\pi_\varphi} \subseteq \mathfrak{B}^{\infty}_{\pi_\varphi} \subseteq \mathfrak{Z}_{\pi_\varphi}$$

### 4.9.5 Universal Collections

As mentioned earlier, these relations might prove to be trivial if the representation is a factor. The observables from the last three subsections have been defined relative to specific von Neumann algebras, but these can be collected together in the same way that the universal representation collects all possible representations. For example, let $\mathfrak{B}^{\infty}_{\pi_u} = \bigoplus_{\varphi \in \mathcal{A}_1^{*+}} \mathfrak{B}^{\infty}_{\pi_\varphi}$ and call this the *universal algebra of observables at infinity*. Since each $\mathfrak{B}^{\infty}_{\pi_\varphi}$ is a subset of $\mathfrak{Z}_{\pi_\varphi}$ which is itself a subset of the center of the universal representation $\mathfrak{Z}_{\pi_u}$, it follows that $\mathfrak{B}^{\infty}_{\pi_u} \subseteq \mathfrak{Z}_{\pi_u}$. Similarly, the *universal set of macroscopic observables* $\mathfrak{M}_{\pi_u} = \bigoplus_{\varphi \in \mathcal{A}_1^{*+}} \mathfrak{M}_{\pi_\varphi}$ and the *universal set of global macroscopic observables* $\mathcal{G}_{\pi_u} = \bigoplus_{\varphi \in \mathcal{A}_1^{*+}} \mathcal{G}_{\pi_\varphi}$ can be defined. Since $\mathcal{G}_{\pi_\varphi} \subseteq \mathfrak{M}_{\pi_\varphi} \subseteq \mathfrak{B}^{\infty}_{\pi_\varphi} \subseteq \mathfrak{Z}_{\pi_\varphi}$ holds for every representation, $\mathcal{G}_{\pi_u} \subseteq \mathfrak{M}_{\pi_u} \subseteq \mathfrak{B}^{\infty}_{\pi_u} \subseteq \mathfrak{Z}_{\pi_u}$. These collections provide the largest possible set of classical, macroscopic, and global macroscopic observables.

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[100] For lattice systems, a global macroscopic observable is defined as: $A_o = \frac{1}{|O|}\sum_{O} A(O)$.

### 4.9.6 HK-Physical Equivalence and Center Observables

Since each of these different types of observables belong to the center of $\pi_u(\mathcal{A})''$, they will have abstract counterparts in $\mathcal{A}^{**}$. The abstract counterparts will be elements of $\mathfrak{Z}(\mathcal{A}^{**})$, so corollary 4.8.3 can be used to show that two primary states are HK-physically equivalent if and only if they have the same expectation values for all of their center observables, observables at infinity, macroscopic observables, and globally macroscopic observables.[101] Thus, whether one agrees with Primas that the classical observables are elements of the center or with Hepp that they are elements of the algebra of observables at infinity or a different subset of the center of $\pi_u(\mathcal{A})''$, corollary 4.8.3 shows that *HK-physical equivalence should more appropriately be considered a criterion for the classical equivalence of two W*-representations which are factors*. HK-physical equivalence is too weak to show that UIRs are physically equivalent as W*-representations and that it is too strong to capture IC-physical equivalence. It also shows that HK-physical equivalence is an appropriate criterion for when two factor representations are *macroscopically* or *globally macroscopically equivalent*.

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[101] Hepp (1972, 242) proved a result similar to theorem 4.8.2. Let $\omega_1$ and $\omega_2$ be primary states and $A_n \in \mathcal{A}(O_n)$. If $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \omega_i(A_n) = a_i$ for $i$ = 1,2 and $a_1 \neq a_2$, then $\omega_1$ and $\omega_2$ are disjoint.

Primary states are dispersion-free for all classical and macroscopic observables in $\mathfrak{Z}\left(\mathcal{A}^{**}\right)$. This follows from condition (2) of lemma 4.8.1. Let *A*=*Z* in condition (3). This implies that $\tilde{\varphi}\left(ZZ\right)=\tilde{\varphi}\left(Z\right)\tilde{\varphi}\left(Z\right)$ for any primary state $\varphi$. Since irreducible representations are factors and are generated from pure abstract states, all pure abstract states are dispersion-free for all classical and macroscopic observables. Pure and primary states take definite values for every observable in $\mathfrak{Z}\left(\mathcal{A}^{**}\right)$. Further, all states in the folium of a representation will have the same definite value for every classical and macroscopic observable. A classical or macroscopic observable is able to classify the disjoint folia since each folium will have different definite values for that observable and that observable will belong to the center of the W*-algebra $\mathcal{A}^{**}$ - not the C*-algebra $\mathcal{A}$. Thus, surprisingly, it is classical-type properties – *not quantum properties* – that distinguish different UIRs.

### 4.10 BUILDING CLASSICAL OBSERVABLES

Up to this point classical observables have been discussed with respect to their ability to discriminate between different UIRs, but their specific form has not been examined. The question is whether classical observables such as temperature and chemical potential should be treated as parameters or non-trivial observables. In many algebraic papers, temperature and chemical potential are treated as mere parameters instead of full-fledged observables. The following argument shows that this option is conceptually incoherent.

If the parameter option is taken, then these quantities correspond to a scalar multiplying the identity operator. To make this more specific let $\varphi$ and $\psi$ be $(\alpha_t, \beta, \mu)$ and $(\alpha_t, \beta', \mu')$-KMS states respectively and assume $\beta \neq \beta'$ and $\mu \neq \mu'$. The temperature observable for $\pi_\varphi(\mathcal{A})''$ ($\pi_\psi(\mathcal{A})''$) is $T_{\pi_\varphi(\mathcal{A})''} = \beta I_{\pi_\varphi}$ ($T_{\pi_\psi(\mathcal{A})''} = \beta' I_{\pi_\psi}$) while the chemical potential observable will be $C_{\pi_\varphi(\mathcal{A})''} = \mu I_{\pi_\varphi}$ ($C_{\pi_\psi(\mathcal{A})''} = \mu' I_{\pi_\psi}$). Now consider the direct sum of the representations $\pi_\omega(\mathcal{A}) = \pi_\varphi(\mathcal{A}) \oplus \pi_\psi(\mathcal{A})$. It will have temperature observable $T_{\pi_\omega(\mathcal{A})''} = \beta I_{\pi_\varphi} \oplus \beta' I_{\pi_\psi}$ and chemical potential observable $C_{\pi_\omega(\mathcal{A})''} = \mu I_{\pi_\varphi} \oplus \mu' I_{\pi_\psi}$. But if $\beta = \mu$ and $\beta' = \mu'$, then $C_{\pi_\omega(\mathcal{A})''} = \left(\beta I_{\pi_\varphi} \oplus \beta' I_{\pi_\psi}\right)$ and $C_{\pi_\omega(\mathcal{A})''} = T_{\pi_\omega(\mathcal{A})''}$. Thus, the temperature and chemical potential observables would be *exactly* the same physical quantity. This is clearly unacceptable since they are different physical quantities. Temperature and chemical potential are directly observable quantities, so they should be represented by some self-adjoint operator on a Hilbert space.

How should these observables be constructed? The following is the first explicit construction of such observables that I am aware of. Each normal state $\tilde{\varphi}$ on $\mathcal{A}^{**}$ has a central projection $C_{\tilde{\varphi}}$ associated with it where $C_{\tilde{\varphi}} \in \mathfrak{Z}(\mathcal{A}^{**})$ and $\tilde{\varphi}(C_{\tilde{\varphi}}) = 1$. By theorem 4.7.6.1, for any two $(\alpha_t, \beta, \mu_j)$ and $(\alpha_t, \gamma, \mu_j)$-KMS states with different temperatures the corresponding representations will be disjoint. Since there is a continuum of disjoint KMS states with different temperatures, there is a continuum of central projections. Each of these central projections corresponds to the yes-or-no question, “Do you have this temperature?” Suppose that $\pi_\varphi$ and $\pi_\psi$ are disjoint. By condition (2) of theorem 4.7.2.1, $C_{\tilde{\varphi}}C_{\tilde{\psi}} = 0 = C_{\tilde{\psi}}C_{\tilde{\varphi}}$. Consider the expectation value $\tilde{\varphi}\left(C_{\tilde{\varphi}}C_{\tilde{\psi}}\right)$. Since the representations are disjoint $\tilde{\varphi}\left(C_{\tilde{\varphi}}C_{\tilde{\psi}}\right) = \tilde{\varphi}(0) = 0$. Since $\pi_\varphi$ and $\pi_\psi$ are type III factor representations and $C_{\tilde{\varphi}}, C_{\tilde{\psi}} \in \mathfrak{Z}\left(\mathcal{A}^{**}\right)$, lemma 4.8.1 implies that $\tilde{\varphi}\left(C_{\tilde{\varphi}}C_{\tilde{\psi}}\right) = \tilde{\varphi}\left(C_{\tilde{\varphi}}\right)\tilde{\varphi}\left(C_{\tilde{\psi}}\right)$ and since $\tilde{\varphi}\left(C_{\tilde{\varphi}}\right) = 1$ this implies that $0 = \tilde{\varphi}(0) = \tilde{\varphi}\left(C_{\tilde{\varphi}}C_{\tilde{\psi}}\right) = \tilde{\varphi}\left(C_{\tilde{\varphi}}\right)\tilde{\varphi}\left(C_{\tilde{\psi}}\right) = \tilde{\varphi}\left(C_{\tilde{\psi}}\right)$. Thus, $\tilde{\varphi}\left(C_{\tilde{\psi}}\right) = 0$ and this implies that $C_{\tilde{\psi}}$ belongs to the kernel of $\pi^w_{\tilde{\varphi}}$. Similar reasoning shows that $C_{\tilde{\varphi}}$ belongs to the kernel of $\pi^w_{\tilde{\psi}}$. Since $C_{\tilde{\psi}} \in \ker \pi^w_{\tilde{\varphi}}$, $C_{\tilde{\varphi}} \in \ker \pi^w_{\tilde{\psi}}$, and $C_{\tilde{\varphi}} \neq 0 \neq C_{\tilde{\psi}}$, it follows that $\ker \pi^w_{\tilde{\varphi}} \neq \ker \pi^w_{\tilde{\psi}}$ and neither $\pi^w_{\tilde{\varphi}}$ nor $\pi^w_{\tilde{\psi}}$ are faithful representations of $\mathcal{A}^{**}$. These facts should not be particularly surprising since $\pi^w_{\tilde{\varphi}}$ and $\pi^w_{\tilde{\psi}}$ are not physically equivalent. Nor should it be surprising that $\tilde{\varphi}\left(C_{\tilde{\psi}}\right) = 0$ since this expectation value corresponds to the question, “what is the probability that $\varphi$ has the same temperature as $\psi$?” This further illuminates the discussion at the end of section 4.6 about the failure of HK-physical equivalence for UIRs, namely that when two W*-representations are not HK-physically equivalent their kernels are not equal to each other. Further, disjoint W*-representations are not faithful representations of $\mathcal{A}^{**}$. In fact, their expectation values will *maximally* differ with respect to $C_{\tilde{\varphi}}$ and $C_{\tilde{\psi}}$: $\tilde{\varphi}\left(C_{\tilde{\psi}}\right)=0$ and $\tilde{\varphi}\left(C_{\tilde{\varphi}}\right)=1$ (similarly, $\tilde{\psi}\left(C_{\tilde{\varphi}}\right)=0$ and $\tilde{\psi}\left(C_{\tilde{\psi}}\right)=1$).

Let us consider the construction of a temperature observable at the concrete level in terms of the direct sum of representations. Since there is a continuum of such states, a direct sum is constructed $\pi_{\omega}=\bigoplus_{\varphi}\pi_{\varphi}$, where each $\pi_{\varphi}$ is a $\left(\alpha_t,\beta,\mu_j\right)$-KMS representation with a different nonzero finite temperature T. Each $\pi_{\varphi}$ is disjoint from every other representation in the direct sum. Since each $\pi_{\varphi}$ is disjoint, by theorem 10.3.5 of (Kadison and Ringrose 1997b) the von Neumann algebra $\pi_{\omega}\left(\mathcal{A}\right)''$ is equal to the direct sum of $\pi_{\varphi}\left(\mathcal{A}\right)''$ von Neumann algebras, i.e. $\pi_{\omega}\left(\mathcal{A}\right)''=\bigoplus_{\varphi}\pi_{\varphi}\left(\mathcal{A}\right)''$ or equivalently $\pi_{\tilde{\omega}}^{w}\left(\mathcal{A}^{**}\right)=\bigoplus_{\tilde{\varphi}}\pi_{\tilde{\varphi}}^{w}\left(\mathcal{A}^{**}\right)$. There is a projection operator $\pi_{\tilde{\varphi}}^{w}\left(C_{\tilde{\varphi}}\right)$ onto each $\pi_{\tilde{\varphi}}^{w}\left(\mathcal{A}^{**}\right)$ and these projection operators will be in the center of $\pi_{\tilde{\omega}}^{w}\left(\mathcal{A}^{**}\right)$, i.e., $\pi_{\tilde{\varphi}}^{w}\left(C_{\tilde{\varphi}}\right)\in\mathfrak{Z}_{\pi_{\tilde{\omega}}^{w}\left(\mathcal{A}^{**}\right)}$. A concrete temperature observable can be constructed from these projection operators by integrating all of these projection operators together. The problem is that integrating requires a measure to integrate with respect to. What measure should be chosen?

The answer comes from the central decomposition theorem. Roughly, every von Neumann algebra can be decomposed into an integral of factors and this is done with respect to a probability measure defined on the center of that von Neumann algebra called the central measure.[102] The central measure gives the probability distribution of the values of the center observables. These observables are the invariants of the system that take specific values. Smaller subcentral decompositions can be introduced, for example, for the algebra of observables at infinity (Bratteli and Robinson 1987, 370). $\pi^{w}_{\tilde{\omega}}\left(\mathcal{A}^{**}\right)=\int^{\oplus}\pi^{w}_{\tilde{\varphi}}\left(\mathcal{A}^{**}\right)d\mu$ has central decomposition, where $\mu$ is the central measure. The concrete temperature observable can be defined as $T_{\pi^{w}_{\tilde{\omega}}}=\int^{\oplus}\lambda_{\tilde{\varphi}}\,\pi^{w}_{\tilde{\varphi}}\left(C_{\tilde{\varphi}}\right)d\mu$, where $\lambda$ is the temperature value associated with each $\pi^{w}_{\tilde{\varphi}}\left(C_{\tilde{\varphi}}\right)$. Notice that a continuum of different disjoint representations has been used to create the temperature observable. Each representation contributed a projection operator to the construction of the temperature observable. Creating classical or macroscopic observables in this way truly puts UIRs to work. A similar construction can be used to build a chemical potential observable.[103]

An abstract temperature observable *T* can also be constructed by integrating all of the central projections associated with each $\left(\alpha_t,\beta,\mu_j\right)$-KMS

[102] For a rigorous discussion of these decompositions see section 4.2.2 of (Bratteli and Robinson 1987).
[103] A chemical potential observable $\mu_j$ can be constructed for each particle species and integrated to create a chemical potential observable for all particle species.

state $\omega$ with a different temperature: $T = \int \lambda_{\tilde{\omega}} C_{\tilde{\omega}} \, d\mu$, where $\lambda_{\tilde{\omega}}$ is the temperature value associated with $\omega$ and the measure is $d\mu$. But what measure should be used in this integration? Every abstract state has its own central measure (see section 14.3 of (Bing-Ren 1992)). Roughly, the central measure of each state is unique and corresponds to its central measure at the concrete level (for more details, see proposition 14.3.4 of (Bing-Ren 1992)). Given how every element in $\pi_u(\mathcal{A})''$ has an abstract counterpart in $\mathcal{A}^{**}$, it should not be shocking that a temperature observable can be constructed at both the concrete and abstract level for W*-algebras.

This construction also solves the question about finding a sufficient condition for classical observables. As discussed in section 4.9, belonging to the center of a von Neumann algebra is necessary but not sufficient for an observable to be considered classical. The question becomes whether belonging to the center can be made a sufficient condition. By constructing a von Neumann algebra using disjoint factor representations, it can. Take the von Neumann algebra used to construct the temperature observable. Each $\pi_\varphi(\mathcal{A})''$ is a factor so none of these von Neumann algebras considered in isolation has any nontrivial observables in their centers. The only nontrivial center elements in the large von Neumann algebra $\pi_\omega(\mathcal{A})''$ are the projectors onto each $\pi_\varphi(\mathcal{A})''$ which make up the concrete temperature observable. Thus, belonging to the center of $\pi_\omega(\mathcal{A})''$ is a sufficient condition for being a classical observable because it was built that way! Constructions of this type permit the construction of only the classical observables that one is interested in. It also allows for the construction of specific types of classical equivalence. Construct a von Neumann algebra $\pi_{\mu T}(\mathcal{A})''$ that only contains the projections for temperature and chemical potential. Call two W*-representations *thermodynamically equivalent* if they are have the same temperature and chemical potential. Two W*-representations which are factors are then thermodynamically equivalent if and only if they are HK-physically equivalent.

## 4.11 The Global Nature of Classical Observables

The case of temperature and chemical potential have a continuum of eigenvalues, but there are other classical observables that have only a discrete number of eigenvalues such as electric charge. For cases like electric charge, there will be at most a countable number of projections. An electric charge observable would be constructed in a similar manner to the temperature observable except the integral would be replaced by a summation. Temperature and chemical potential generate continuous superselection sectors where the representation associated with each sector has a different temperature or chemical potential and the representations associated with two different sectors are disjoint from each other. Electric charge generates discrete superselection sectors where each sector is associated with a representation that has a different integer value for electric charge.

The following argument by Robinson (1966, 484) explains why many classical observables are global quantities, and, as such, do not belong to a C*-algebra of local or quasi-local observables. Let **S** be some finite region of three dimensional spacetime at a fixed time, $\mathcal{R}(\mathbf{S})$ a local von Neumann algebra, and the total charge is: $Q_{tot} = Q_{int} + Q_{ext}$, where $Q_{int}$ is the total charge inside **S** and $Q_{ext}$ is the total charge outside of **S**. Since the total charge is conserved, $\left[Q_{tot}, \mathcal{R}(\mathbf{S})\right] = 0$. One of the axioms of AQFT is that the algebra of observables defined on two spacelike regions commute, i.e., their commutator is zero. Since $Q_{ext}$ is spacelike separated from **S** by definition, $\left[Q_{ext}, \mathcal{R}(\mathbf{S})\right] = 0$. $\left[Q_{tot}, \mathcal{R}(\mathbf{S})\right] = 0$ and $\left[Q_{ext}, \mathcal{R}(\mathbf{S})\right] = 0$ imply that $\left[Q_{int}, \mathcal{R}(\mathbf{S})\right] = 0$. This implies that $Q_{int}$ has to belong to the commutant of $\mathcal{R}(\mathbf{S})$. By another assumption called duality,[104] $Q_{int} \in \left\{\mathcal{R}(\mathbf{S})\right\}'$ implies that $Q_{int}$ belongs to the algebra of observables defined on the region which is the causal complement of **S**, i.e., $Q_{int} \in \mathcal{R}(\mathbf{S}')$. Thus, the total charge inside a given volume of space can be determined by measurements outside of this volume. As Robinson noted, this is the import of Gauss' Law in electrostatics. Another observable that would fall into this category would be the total energy (Haag and Kastler 1964, 849). Notice that for an observable to be a global observable, using Robinson's argument, three

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[104] Duality is the assumption that $\mathcal{R}(\mathbf{S}') = \left\{\mathcal{R}(\mathbf{S})\right\}'$, where $\mathbf{S}'$ is the causal complement to **S** (i.e., the set of points spacelike separated from **S**) and $\left\{\mathcal{R}(\mathbf{S})\right\}'$ is the commutant of $\mathcal{R}(\mathbf{S})$.

conditions must be met: (1) the observable has a conservation law associated with it, (2) local commutativity must hold, and (3) duality must be assumed. While (2) and (3) are fairly benign assumptions, not all classical observables have conservation laws associated with them, e.g., temperature.[105]

### 4.12 The Unruh Effect

Now that the appropriate mathematical tools have been laid out, they will be put to use in the case of the Unruh effect. The Unruh effect was briefly discussed in chapter three in the context of canonical QFT, but it can be rigorously formulated using the algebraic approach. The Weyl form of the CCRs determines the C*-algebra known as the Weyl algebra $\mathcal{W}$.[106] The Unruh effect is generated by using Rindler coordinates in Minkowski spacetime.[107] These coordinates are only defined on the Rindler wedges shown in the diagram below.

[105] One might argue that since temperature can be viewed as a measure of the mean kinetic energy of the atoms, conservation of energy would be the appropriate conservation law. However, the kinetic energy is in general not conserved; only the total kinetic and potential energy is conserved.
[106] For details, see (Clifton and Halvorson 2001).
[107] For details, see section 2 of (Arageorgis, Earman, and Ruetsche 2002a).

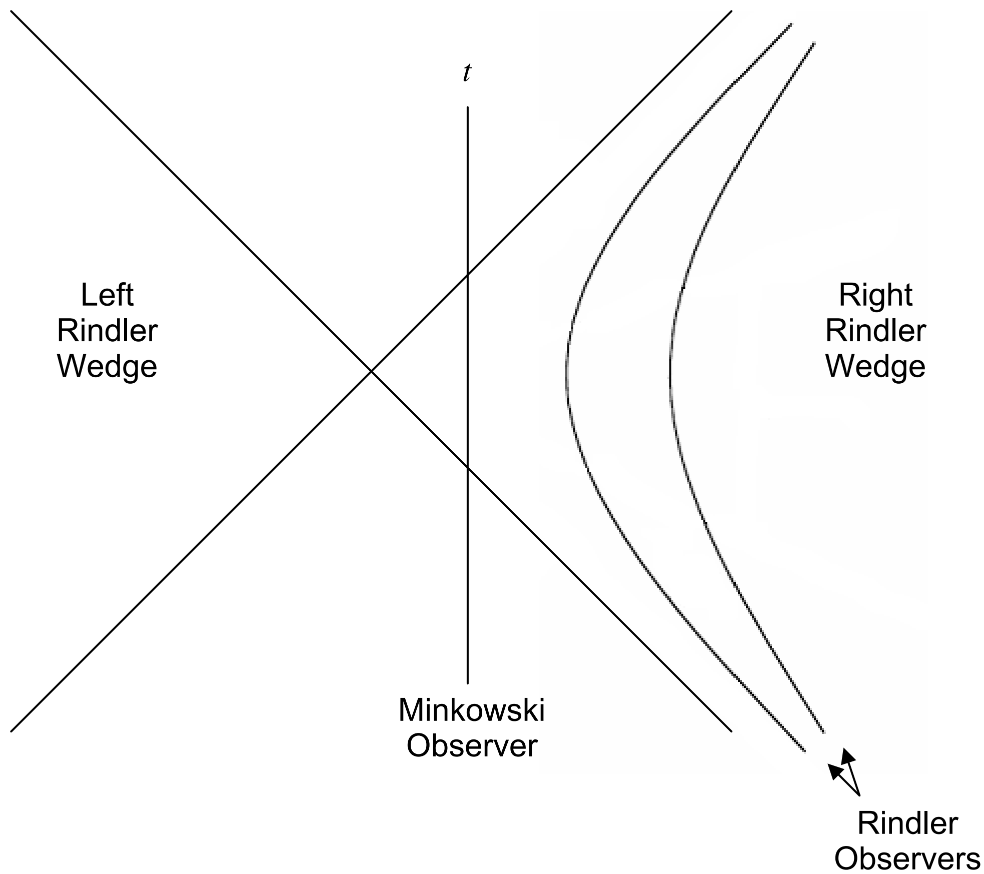


Figure 4.2: The Motion of Minkowski and Rindler Observers

These wedges have two types of trajectories defined on them: inertial motion and uniformly accelerated motion. The Minkowski observer in, say, the right Rindler frame is uniformly accelerating while a Rindler observer's trajectory is inertial. Each observer will associate an algebra of local observables with the right wedge and build a representation via the GNS theorem.[108] The Rindler representation $\pi_{\omega_R^{\triangleleft}}$ is only defined on the right wedge and is generated from its Rindler vacuum state $\omega_R^{\triangleleft}$; it is not defined on all of Minkowski spacetime. When the Minkowski vacuum state $\omega_{\mathcal{M}}$ is restricted to the right Rindler wedge $\omega_{\mathcal{M}}^{\triangleleft}$, it is a mixed state whereas $\omega_R^{\triangleleft}$ is a pure state. Thus, the Minkowski representation $\pi_{\omega_{\mathcal{M}}^{\triangleleft}}$ restricted to the right Rindler wedge is a reducible representation while the Rindler representation $\pi_{\omega_R^{\triangleleft}}$ on the right Rindler wedge is irreducible. Further, $\pi_{\omega_R^{\triangleleft}}$ is a type I factor while $\pi_{\omega_{\mathcal{M}}^{\triangleleft}}$ is a type III factor. Clifton and Halvorson (2001, 463) proved the following theorem.

**Theorem 4.12.1:** $\pi_{\omega_{\mathcal{M}}^{\triangleleft}}$ and $\pi_{\omega_R^{\triangleleft}}$ are disjoint.

Since both $\pi_{\omega_R^{\triangleleft}}$ and $\pi_{\omega_{\mathcal{M}}^{\triangleleft}}$ are factors and disjoint, they are not quasi-equivalent by theorem 4.7.3.1. The following corollary then follows immediately from theorem 4.6.2.

**Corollary 4.12.2:** $\pi^{w}_{\tilde{\omega}_{\mathcal{M}}^{\triangleleft}}$ and $\pi^{w}_{\tilde{\omega}_R^{\triangleleft}}$ are not HK-physically equivalent.

Thus, the Unruh effect is an example in AQFT of two HK-physically inequivalent W*-representations.

[108] For simplicity, only the right Rindler wedge will be considered below, however, the results apply to the left Rindler wedge and the algebra of observables defined on both the left and right wedges; see (Clifton and Halvorson 2001) for details.

The explanation of their HK-physical *inequivalence* can be deepened. Since both representations are factors and they are HK-physically inequivalent, the following corollary immediately follows from corollary 4.8.3.

**Corollary 4.12.3:** There exists an observable $Z \in \mathfrak{Z}\left(\mathcal{W}_{\triangleleft}^{**}\right)$ in the center of the bidual of the Weyl algebra on the right Rindler wedge $\mathcal{W}_{\triangleleft}^{**}$, such that $\tilde{\omega}_{R}^{\triangleleft}(Z) \neq \tilde{\omega}_{\mathcal{M}}^{\triangleleft}(Z)$.

What physical quantity is *Z*? The answer is found by using the concepts from both AQFT and AQSM. The Minkowski vacuum state $\omega_{\mathcal{M}}^{\triangleleft}$, when restricted to the right wedge, satisfies the KMS condition (Arageorgis, Earman, and Ruetsche 2002a, 188-189). It is a thermal state and it has a nonzero finite temperature. More specifically, $\omega_{\mathcal{M}}^{\triangleleft}$ is a KMS state at temperature $T = 1/2\pi$ with respect to the automorphism group $\alpha_{\eta}$ where $\eta$ is the Rindler time (part (ii) of lemma 3 (Arageorgis, Earman, and Ruetsche 2002a, 188).

The situation can also be expressed in the following way (Arageorgis, Earman, and Ruetsche 2002a, 189). Let $\omega$ be a $(\alpha_t, \beta, \mu)$-KMS state. $\omega$ is a KMS state at inverse temperature $\beta/C$ with respect to $\alpha_{t'}$ where $t' = t/C$, if *C* is a positive constant. Let *a* be the magnitude of the acceleration along a particular Rindler trajectory. The proper time along that trajectory $\tau_a$ is related to the Rindler time $\eta$ by $\tau_a = \eta/a$. In this case, there are KMS states, one for each acceleration $a \in (0,+\infty)$ at temperature $a/2\pi$ with respect to the automorphism group $\alpha_{\tau_a}$. Thus, one can say either that $\omega_{\mathcal{M}}^{\triangleleft}$ is a KMS state at temperature $T = 1/2\pi$ with respect to the automorphism group $\alpha_\eta$ or that there is a continuum of $\omega_{\mathcal{M}}^{\triangleleft}$ KMS states for each acceleration $a \in (0,+\infty)$ at temperature $T = a/2\pi$ with respect to the automorphism group $\alpha_{\tau_a}$. The following corollary can now be proven.

**Corollary 4.12.4:** Let $\omega_{\mathcal{M},a}^{\triangleleft}$ be a $(\alpha_{\tau_a}, \beta, \mu)$-KMS state at temperature $T = a/2\pi$ with acceleration *a*, where $a \in (0,+\infty)$, and $\omega_{\mathcal{M},a'}^{\triangleleft}$ be a $(\alpha_{\tau_{a'}}, \beta', \mu)$-KMS state at temperature $T' = a'/2\pi$, where $a' \in (0,+\infty)$. Assume $a \neq a'$. Then $\pi_{\tilde{\omega}_{\mathcal{M},a}^{\triangleleft}}^{w}$ and $\pi_{\tilde{\omega}_{\mathcal{M},a'}^{\triangleleft}}^{w}$ are not HK-physically equivalent.

Proof: Since both $\pi_{\tilde{\omega}_{\mathcal{M},a}^{\triangleleft}}^{w}$ and $\pi_{\tilde{\omega}_{\mathcal{M},a'}^{\triangleleft}}^{w}$ are KMS states at nonzero finite temperatures, both are type III factors. Since they have different accelerations, and hence different temperatures, theorem 4.7.6.1 implies that they are disjoint. Since they are disjoint and factors, they are not quasi-equivalent by theorem 4.7.3.1. It then follows from theorem 4.6.2 that $\pi_{\tilde{\omega}_{\mathcal{M},a}^{\triangleleft}}^{w}$ and $\pi_{\tilde{\omega}_{\mathcal{M},a'}^{\triangleleft}}^{w}$ are *not* HK-physically equivalent.

This corollary shows not only that there exists a continuum of UIRs in the Unruh effect, but that they are each HK-physically *inequivalent* to each other and $\pi_{\tilde{\omega}_{R}^{\triangleleft}}^{w}$!

The discussion of the Unruh effect normally only focuses on $\pi_{\omega_{\mathcal{M}}^{\triangleleft}}$ and $\pi_{\omega_{R}^{\triangleleft}}$. The fact that the Minkowski and Rindler representations are disjoint is not terribly surprising since they are different factor types. But each $\pi^{w}_{\tilde{\omega}^{\triangleleft}_{\mathcal{M},a}}$ is a type III factor and the fact that different type III factors with infinitesimally different accelerations generates UIRs *is* surprising. From corollary 4.8.3 and corollary 4.12.4, there exists at least one $Z \in \mathfrak{Z}\left(\mathcal{W}_{\triangleleft}^{**}\right)$ that distinguishes each $\pi^{w}_{\tilde{\omega}^{\triangleleft}_{\mathcal{M},a}}$.

> **Corollary 4.12.5:** Let $\omega^{\triangleleft}_{\mathcal{M},a}$ be a $\left(\alpha_{\tau_a}, \beta, \mu\right)$-KMS state at temperature $T = a/2\pi$ with acceleration *a*, where $a \in (0,+\infty)$, and $\omega^{\triangleleft}_{\mathcal{M},a'}$ be a $\left(\alpha_{\tau_{a'}}, \beta', \mu\right)$-KMS state at temperature $T' = a'/2\pi$, where $a' \in (0,+\infty)$. Assume $a \neq a'$. Then there exists a $Z \in \mathfrak{Z}\left(\mathcal{W}_{\triangleleft}^{**}\right)$ such that $\tilde{\omega}^{\triangleleft}_{\mathcal{M},a}(Z) \neq \tilde{\omega}^{\triangleleft}_{\mathcal{M},a'}(Z)$.

It is now obvious what physical quantity *Z* represents that distinguishes these different representations in the Unruh effect: it is temperature. Using the construction technique for classical observables discussed in section 4.10, a temperature observable *T* can be constructed in the Unruh effect by direct summing all of the projection operators associated with the von Neumann algebras generated by each $\omega^{\triangleleft}_{\mathcal{M},a}$ with a different temperature $\lambda$;

$T_{\pi^{w}_{\tilde{\omega}^{\triangleleft}_{\mathcal{M}}}} = \int^{\oplus} \lambda_{\tilde{\omega}^{\triangleleft}_{\mathcal{M},a}} \, \pi^{w}_{\tilde{\omega}^{\triangleleft}_{\mathcal{M},a}}\left(C_{\tilde{\omega}^{\triangleleft}_{\mathcal{M},a}}\right) d\mu$. Since the acceleration is proportional to the temperature via $a = 2\pi T$, an acceleration observable can be constructed in the same way $a = 2\pi T = 2\pi \int^{\oplus} \lambda_{\tilde{\omega}^{\triangleleft}_{\mathcal{M},a}} \, \pi^{w}_{\tilde{\omega}^{\triangleleft}_{\mathcal{M},a}} \left( C_{\tilde{\omega}^{\triangleleft}_{\mathcal{M},a}} \right) d\mu$.

### 4.13 Conclusion

The algebraic approach to QFT allows the issue of UIRs to examined in rigorous mathematical detail. It also offers a technical argument against the physical significance of UIRs using Haag and Kastler's notion of physical equivalence. However, this "solution" is fatally flawed. Most of the representations used in AQFT are W*-representations. For two W*-representations to be physically equivalent in the Haag and Kastler sense requires in many cases (irreducible representations, type III factors, KMS states) that their C*-representations be unitarily equivalent. So, mere unitary inequivalence is sufficient to render these W*-representations HK-physically *inequivalent*. However, HK-physical equivalence can be put to use as a criteria for when two W*-representations are classically equivalent. The theorems proved in this chapter show that in many cases the difference between UIRs that are HK-physically inequivalent is that they differ in their expectation values for a classical observable such as temperature or chemical potential. These observables are not part of the original C*-algebra, but rather a larger W*-algebra called the bidual $\mathcal{A}^{**}$. This was illustrated in the case of the Unruh effect. The lesson of UIRs is that the C*-algebra was too small to be able to capture all of the possible physical content of AQFT and AQSM. The debate about where to locate the physical content of AQFT and AQSM will be examined in the next chapter.

## 4.14 PROOFS

**Theorem 4.7.7:** Let $\varphi$, $\psi$ be KMS states. $\pi_\varphi(\mathcal{A})$ and $\pi_\psi(\mathcal{A})$ are quasi-equivalent if and only if they are unitarily equivalent.

Proof: Unitary equivalence always implies quasi-equivalence, so the only part left to prove is that quasi-equivalence implies unitary equivalence. The theorem that will help prove this is the unitary implementation theorem proved by Kadison and Ringrose (1997a, 469-470).

Unitary Implementation Theorem: If $\alpha$ is a *-isomorphism of the von Neumann algebra $\mathcal{R}_1$, with unit separating and cyclic vector $x$, onto the von Neumann algebra $\mathcal{R}_2$, with unit separating and cyclic vector $y$, then there is a unitary transformation $U$ of the Hilbert Space $\mathcal{H}_1$, on which $\mathcal{R}_1$ acts, onto the Hilbert space $\mathcal{H}_2$, on which $\mathcal{R}_2$ acts, such that $\alpha(A) = UAU^{-1}$ for all $A \in \mathcal{R}_1$.

Recall that two representations $\pi_\varphi$, $\pi_\psi$ of a C*-algebra $\mathcal{A}$ are quasi-equivalent if and only if there is a *-isomorphism $\alpha : \pi_\varphi(\mathcal{A})'' \to \pi_\psi(\mathcal{A})''$ such that

$\pi_\psi(\mathcal{A}) = \alpha(\pi_\varphi(\mathcal{A}))$ for all $A \in \mathcal{A}$. Thus, the unitary implementation theorem can be used to conclude that quasi-equivalence implies unitary equivalence if it can be shown that both von Neumann algebras have a separating and cyclic vector. (A unit separating (cyclic) vector can always be formed from a separating (cyclic) vector by normalizing it, i.e. dividing by its length.) For any KMS state $\omega$, Winnink (1970) proved that $\pi_\omega(\mathcal{A})''$ has a cyclic and separating vector. Thus, since $\pi_\varphi(\mathcal{A})''$ and $\pi_\psi(\mathcal{A})''$ are assumed to be quasi-equivalent and they each have cyclic and separating vectors, they are unitarily equivalent by the unitary implementation theorem.

**Theorem 4.7.8:** Let $\pi_\varphi$ and $\pi_\psi$ be type III factor representations of a C*-algebra $\mathcal{A}$ into the bounded operators $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is separable. $\pi_\varphi$ and $\pi_\psi$ are quasi-equivalent if and only if they are unitarily equivalent.

Proof: The unitary implementation theorem will again be used to show that quasi-equivalence implies unitary equivalence. To use this theorem, it must be shown that type III factors have a cyclic and separating vector. By proposition 9.1.6 (Kadison and Ringrose 1997a, 590), if $\mathcal{R}$ is a countably decomposable, properly infinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, then $\mathcal{R}$ has a joint cyclic and separating vector. Thus, if it can be shown that type III factors and their commutant are both properly infinite and countably decomposable, then proposition 9.1.6 shows that they have a joint cyclic and separating vector which allows the unitary implementation theorem to be used. Type III factors are properly infinite by definition (Kadison and Ringrose 1997a, 411) and since the commutant of a type III factor is itself type III (see theorem 9.1.3 (Kadison and Ringrose 1997a, 588)) it follows that the representation is also properly infinite. All that remains is to show that $\mathcal{R}_1$ and $\mathcal{R}_2$ are countably decomposable. By the definition and discussion of countably decomposable von Neumann algebras (Kadison and Ringrose 1997a, 338) and the assumption that $\mathcal{H}$ is separable, it follows that $\pi_\varphi(\mathcal{A})''$ and $\pi_\psi(\mathcal{A})''$ are countably decomposable, which completes the proof.

**Lemma 4.8.1:** Let $\varphi$ be a state on the C*-algebra $\mathcal{A}$, $\tilde{\varphi}$ the extension of $\varphi$ to be a normal state on the W*-algebra $\mathcal{A}^{**}$, and denote $\mathfrak{Z}(\mathcal{A}^{**})$ as the center of $\mathcal{A}^{**}$. Then each condition implies the condition below it..

(1) $\varphi$ is primary
(2) $\pi_{\tilde{\varphi}}(Z) = \tilde{\varphi}(Z)\mathrm{I}_{\pi^w_{\tilde{\varphi}}}$ where $\mathrm{I}_{\pi^w_{\tilde{\varphi}}}$ is the identity operator in $\mathcal{H}_{\pi^w_{\tilde{\varphi}}}$ and $Z \in \mathfrak{Z}(\mathcal{A}^{**})$
(3) $\tilde{\varphi}(AZ) = \tilde{\varphi}(A)\tilde{\varphi}(Z) = \varphi(A)\tilde{\varphi}(Z)$ for all $A \in \mathcal{A}$ and $Z \in \mathfrak{Z}(\mathcal{A}^{**})$
(4) the restriction of $\tilde{\varphi}$ to $\mathfrak{Z}(\mathcal{A}^{**})$ is pure

Proof:

(1) $\Rightarrow$ (2)

Suppose $\pi_\varphi(\mathcal{A})$ is a factor, which by definition means $\pi_\varphi(\mathcal{A})'' \cap \pi_\varphi(\mathcal{A})' = \lambda \mathrm{I}_{\pi_\varphi}$ where $\lambda \in \mathbb{C}$. For any $Z \in \mathfrak{Z}(\mathcal{A}^{**})$, obviously $\pi^w_{\tilde{\varphi}}(Z) \in \pi^w_{\tilde{\varphi}}(\mathcal{A}^{**}) = \pi_\varphi(\mathcal{A})''$. $\pi^w_{\tilde{\varphi}}(Z) \in \pi_\varphi(\mathcal{A})'''$ since $\pi^w_{\tilde{\varphi}}(Z)$ commutes with everything in $\pi_\varphi(\mathcal{A})''$. But since

$\pi_{\varphi}(\mathcal{A})'''$ is a von Neumann algebra it is equal to $\pi_{\varphi}(\mathcal{A})'$, so $\pi_{\tilde{\varphi}}^{w}(Z) \in \pi_{\varphi}(\mathcal{A})'$.

Thus, since $\pi_{\tilde{\varphi}}^{w}(Z) \in \pi_{\varphi}(\mathcal{A})''$, $\pi_{\tilde{\varphi}}^{w}(Z) \in \pi_{\varphi}(\mathcal{A})'$, and $\pi_{\varphi}(\mathcal{A})'' \cap \pi_{\varphi}(\mathcal{A})' = \lambda \mathrm{I}_{\pi_{\varphi}}$,

$\pi_{\tilde{\varphi}}^{w}(Z) = \lambda \mathrm{I}_{\pi_{\tilde{\varphi}}}$. Now by definition, $\tilde{\varphi}(Z) = \left\langle \Omega_{\pi_{\tilde{\varphi}}^{w}} \middle| \pi_{\tilde{\varphi}}^{w}(Z) \middle| \Omega_{\pi_{\tilde{\varphi}}^{w}} \right\rangle$ and substituting

$\pi_{\tilde{\varphi}}^{w}(Z) = \lambda \mathrm{I}_{\pi_{\tilde{\varphi}}}$ gives $\tilde{\varphi}(Z) = \left\langle \Omega_{\pi_{\tilde{\varphi}}^{w}} \middle| \pi_{\tilde{\varphi}}^{w}(Z) \middle| \Omega_{\pi_{\tilde{\varphi}}^{w}} \right\rangle = \left\langle \Omega_{\pi_{\tilde{\varphi}}^{w}} \middle| \lambda \mathrm{I}_{\pi_{\tilde{\varphi}}^{w}} \middle| \Omega_{\pi_{\tilde{\varphi}}^{w}} \right\rangle = \lambda$ since

$\left\langle \Omega_{\pi_{\tilde{\varphi}}^{w}} \middle| \Omega_{\pi_{\tilde{\varphi}}^{w}} \right\rangle = 1$. Thus, $\pi_{\tilde{\varphi}}^{w}(Z) = \lambda \mathrm{I}_{\pi_{\tilde{\varphi}}^{w}} = \tilde{\varphi}(Z) \mathrm{I}_{\pi_{\tilde{\varphi}}^{w}}$.

(2) $\Rightarrow$ (3)

Assume $\pi_{\tilde{\varphi}}^{w}(Z) = \tilde{\varphi}(Z) \mathrm{I}_{\pi_{\tilde{\varphi}}^{w}}$.

$$\begin{aligned}
\tilde{\varphi}(AZ) &= \left\langle \Omega_{\pi_{\tilde{\varphi}}^{w}} \middle| \pi_{\tilde{\varphi}}^{w}(AZ) \middle| \Omega_{\pi_{\tilde{\varphi}}^{w}} \right\rangle = \left\langle \Omega_{\pi_{\tilde{\varphi}}^{w}} \middle| \pi_{\tilde{\varphi}}^{w}(A) \pi_{\tilde{\varphi}}^{w}(Z) \middle| \Omega_{\pi_{\tilde{\varphi}}^{w}} \right\rangle = \left\langle \Omega_{\pi_{\tilde{\varphi}}^{w}} \middle| \pi_{\tilde{\varphi}}^{w}(A) \tilde{\varphi}(Z) \mathrm{I}_{\pi_{\tilde{\varphi}}^{w}} \middle| \Omega_{\pi_{\tilde{\varphi}}^{w}} \right\rangle \\
&= \left\langle \Omega_{\pi_{\tilde{\varphi}}^{w}} \middle| \pi_{\tilde{\varphi}}^{w}(A) \mathrm{I}_{\pi_{\tilde{\varphi}}^{w}} \middle| \Omega_{\pi_{\tilde{\varphi}}^{w}} \right\rangle \tilde{\varphi}(Z) = \left\langle \Omega_{\pi_{\tilde{\varphi}}^{w}} \middle| \pi_{\tilde{\varphi}}^{w}(A) \middle| \Omega_{\pi_{\tilde{\varphi}}^{w}} \right\rangle \tilde{\varphi}(Z) = \tilde{\varphi}(A) \tilde{\varphi}(Z) = \varphi(A) \tilde{\varphi}(Z)
\end{aligned}$$

The third equality follows from one of the properties ($\pi_{\omega}(AB) = \pi_{\omega}(A)\pi_{\omega}(B)$) of a representation (see section 4.2) and the last equality follows since $\tilde{\varphi}(A) = \varphi(A)$ for all $A \in \mathcal{A}$.

(3) $\Rightarrow$ (4)

Since $\mathfrak{Z}(\mathcal{A}^{**})$ is commutative, it is isomorphic to the algebra of continuous complex valued functions C(X) on some compact Hausdorff space X. The proof is then just a special case of theorem 3.4.7 of (Kadison and Ringrose 1997a, 213-214).

**Theorem 4.8.2:** Let $\varphi$, $\psi$ be two primary states. $\pi_\varphi$ and $\pi_\psi$ are quasi-equivalent if and only if for every observable $Z \in \mathfrak{Z}(\mathcal{A}^{**})$, $\tilde{\varphi}(Z) = \tilde{\psi}(Z)$.[109]

Proof: $(\Leftarrow)$ Assume that $\tilde{\varphi}(Z) = \tilde{\psi}(Z)$ for every $Z \in \mathfrak{Z}(\mathcal{A}^{**})$. Recall from condition (4) of theorem 4.6.1 that two states $\pi_\varphi$, $\pi_\psi$ are *quasi-equivalent* if and only if their enveloping central supports are equal, i.e. $C_{\tilde{\varphi}} = C_{\tilde{\psi}}$. This is the condition that must be satisfied to finish this part of the proof. Let $Z = C_{\tilde{\varphi}}$. Then $\tilde{\varphi}(C_{\tilde{\varphi}}) = \tilde{\psi}(C_{\tilde{\varphi}})$ and since by definition $\tilde{\varphi}(C_{\tilde{\varphi}}) = 1$ this implies that $\tilde{\psi}(C_{\tilde{\varphi}}) = 1$. So, $C_{\tilde{\varphi}}$ could be the enveloping central support of $\psi$ which implies that $C_{\tilde{\psi}} \leq C_{\tilde{\varphi}}$. Now let $Z = C_{\tilde{\psi}}$. Then $\tilde{\varphi}(C_{\tilde{\psi}}) = \tilde{\psi}(C_{\tilde{\psi}})$ and since by definition $\tilde{\psi}(C_{\tilde{\psi}}) = 1$ this implies that $\tilde{\varphi}(C_{\tilde{\psi}}) = 1$. So, $C_{\tilde{\psi}}$ could be the enveloping central support of $\tilde{\varphi}$ which implies that $C_{\tilde{\varphi}} \leq C_{\tilde{\psi}}$. Thus, $C_{\tilde{\varphi}} = C_{\tilde{\psi}}$ which proves that $\pi_\varphi$, $\pi_\psi$ are quasi-equivalent.

[109] The idea for this theorem came from the work of Primas (2000) and Hepp (1972). I gratefully acknowledge the help of Hans Halvorson and Hans Primas in the proof of theorem 4.8.2 as well as Halvorson's help with the proof of lemma 4.8.1.

$(\Rightarrow)$ Let $\pi_\varphi$, $\pi_\psi$ be quasi-equivalent. Then there is an isomorphism between $\pi_\varphi(\mathcal{A})''$ and $\pi_\psi(\mathcal{A})''$ (and which can be extended to be an isomorphism between $\pi^w_{\tilde{\varphi}}(\mathcal{A}^{**})$ and $\pi^w_{\tilde{\psi}}(\mathcal{A}^{**})$). By assumption, $\varphi$ and $\psi$ are primary states and by lemma 4.8.1 this implies that the restrictions of $\tilde{\varphi}$ and $\tilde{\psi}$ to $\mathfrak{Z}(\mathcal{A}^{**})$ are pure. This implies that $\pi^w_{\tilde{\varphi}}\big|\mathfrak{Z}(\mathcal{A}^{**})$ and $\pi^w_{\tilde{\psi}}\big|\mathfrak{Z}(\mathcal{A}^{**})$ are irreducible. Being irreducible implies that the isomorphism between $\pi^w_{\tilde{\varphi}}\big|\mathfrak{Z}(\mathcal{A}^{**})$ and $\pi^w_{\tilde{\psi}}\big|\mathfrak{Z}(\mathcal{A}^{**})$ can be implemented by a unitary operator. Thus, $\pi^w_{\tilde{\varphi}}(Z) = U\pi^w_{\tilde{\psi}}(Z)U^{-1}$for all $Z \in \mathfrak{Z}(\mathcal{A}^{**})$. Lemma 4.8.1 also implies that $\pi^w_{\tilde{\varphi}}(Z) = \tilde{\varphi}(Z)\mathrm{I}_{\pi^w_{\tilde{\varphi}}}$ and $\pi^w_{\tilde{\psi}}(Z) = \tilde{\psi}(Z)\mathrm{I}_{\pi^w_{\tilde{\psi}}}$. Substituting these equations in gives $\tilde{\varphi}(Z)I_{\pi_{\tilde{\varphi}}} = U\tilde{\psi}(Z)\mathrm{I}_{\pi^w_{\tilde{\varphi}}}U^{-1} = \tilde{\psi}(Z)\mathrm{I}_{\pi^w_{\tilde{\psi}}}$. Since $\pi^w_{\tilde{\varphi}}\big|\mathfrak{Z}(\mathcal{A}^{**})$ and $\pi^w_{\tilde{\psi}}\big|\mathfrak{Z}(\mathcal{A}^{**})$are irreducible representations of an abelian algebra, this implies that that the representations are 1-dimensional. In other words, it consists of complex multiples of the identity operator. Thus, $\mathrm{I}_{\pi^w_{\tilde{\varphi}}} = \mathrm{I}_{\pi^w_{\tilde{\psi}}}$ which implies that $\tilde{\varphi}(Z) = \tilde{\psi}(Z)$ for all $Z \in \mathfrak{Z}(\mathcal{A}^{**})$.

**Corollary 4.8.3:** Let $\varphi$, $\psi$ be two primary states. The von Neumann algebras, $\pi_\varphi(\mathcal{A})''$and $\pi_\psi(\mathcal{A})''$, are HK-physically equivalent if and only if for every observable $Z \in \mathfrak{Z}(\mathcal{A}^{**})$, $\tilde{\varphi}(Z) = \tilde{\psi}(Z)$.

Proof:

$(\Rightarrow)$ Since $\pi_{\varphi}(\mathcal{A})''$ and $\pi_{\psi}(\mathcal{A})''$ are assumed to be HK-physically equivalent, they are quasi-equivalent by theorem 4.6.2. Since they are quasi-equivalent, $\tilde{\varphi}(Z) = \tilde{\psi}(Z)$ for every observable $Z \in \mathfrak{Z}(\mathcal{A}^{**})$ by theorem 4.8.2.

$(\Leftarrow)$ Assume $\tilde{\varphi}(Z) = \tilde{\psi}(Z)$ for every observable $Z \in \mathfrak{Z}(\mathcal{A}^{**})$. By theorem 4.8.2 and the assumption that $\pi_{\varphi}(\mathcal{A})''$ and $\pi_{\psi}(\mathcal{A})''$ are factors, it follows that $\pi_{\varphi}(\mathcal{A})''$ and $\pi_{\psi}(\mathcal{A})''$ are quasi-equivalent. By theorem 4.6.2, it then follows that $\pi_{\varphi}(\mathcal{A})''$ and $\pi_{\psi}(\mathcal{A})''$ are HK-physically equivalent.

# Chapter Five

## 5.1 Introduction

The issue about where to locate physical content in AQFT was raised in the last chapter in connection with Haag and Kastler's notion of physical equivalence and algebraic imperialism. However, the technical machinery of HK-physical equivalence is not sufficient to underwrite a strong form of algebraic imperialism. So the question remains: what mathematical structure captures the physical content of UIRs and more generally that of AQFT? This chapter investigates this issue by examining the work of Ruetsche (2002) (2003) (2006) (2007), Kronz and Lupher (2005), and Clifton and Halvorson (2001). Sections 5.1 – 5.8 will discuss different mathematical structures for capturing the physical content of AQFT. I will argue that the bidual $\mathcal{A}^{**}$ is the best mathematical structure for this task in section 5.9. Finally, it has been argued (Arageorgis, Earman, and Ruetsche 2002b) that UIRs should be thought of as incommensurable theories. This argument will be critically evaluated in section 5.10.

Laura Ruetsche has addressed the location of physical content in AQFT in two published papers (2002) (2003) and two draft manuscripts (2006) (2007).[110] For Ruetsche, the content of a physical theory can be specified by a pair (O, S), where O is the set of observables or physical magnitudes and S is the set of possible states of a system. For AQFT, the most natural possible values for (O, S) are the different mathematical structures given in the diagram from the last chapter.

C*-Algebra W*-Algebra

$$
\begin{tikzcd}
\text{Abstract:} & \mathcal{A} \arrow[r, "*"] \arrow[dd, "\pi_\omega"'] & \mathcal{A}^{*} \arrow[r, "*"] & \mathcal{A}^{**} \arrow[dd, "\pi^{w}_{\tilde{\omega}}"'] \\
& & \begin{array}{c} \cup \\ \omega \in \mathcal{A}_1^{*+} \end{array} & \\
\text{Concrete:} & \pi_\omega(\mathcal{A}) \arrow[rr, "{''} \text{ or } -"] & & \pi_\omega(\mathcal{A})'' = \pi^{w}_{\tilde{\omega}}\left(\mathcal{A}^{**}\right) = \pi_\omega(\mathcal{A})^{-}
\end{tikzcd}
$$

In her papers, Ruetsche has argued against two possible positions on where to locate content in the algebraic approach. The terminology of these positions has changed over time, but the basic content has not. Ruetsche (2002, 348) originally discussed two “fictional and extreme interpreters of QFT,” whom she called the *algebraic imperialist* and the *Hilbert space conservative*. As mentioned in the last chapter, the position of algebraic imperialism originated in Arageorgis (1995).[111] In her (2003) paper, she renamed them: algebraic chauvinism and Hilbert space chauvinism, while in Ruetsche’s (2006) paper she discusses the

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110 I would like to thank Laura Ruetsche for sharing these manuscripts with me.

*algebraic package* and the *Hilbert space package*. Her latest work (2007) on the subject returns to the algebraic imperialism and Hilbert space conservative nomenclature.

## 5.2 ALGEBRAIC IMPERIALISM

The position of algebraic imperialism has been discussed quite a bit in the previous chapter, so only a brief summary of the position and its problems will be given here. Algebraic imperialism denies that quantum theories are essentially Hilbert space theories. Support for the position comes from the GNS theorem and Haag and Kastler's notion of physical equivalence. These technical developments are supposed to show that Hilbert spaces and representations are dispensable.[112] According to the algebraic imperialist, the physical content of AQFT is completely contained in the C*-algebra $\mathcal{A}$ and the possible state space for the algebraic imperialist is the set of all algebraic states in $\mathcal{A}_1^{*+}$. Thus, the physical content of QFT for an algebraic imperialist is: $\left(\mathcal{A}, \mathcal{A}_1^{*+}\right)$.

---

[111] The position of the Hilbert space conservative is probably derived from the attitude of physicists that only the Fock representation is necessary for QFT. Chapter three discussed the problems with this *Fock representation chauvinist* position.

[112] Ruetsche (2002, 361) adds an additional argument for algebraic imperialism. All representations of the Weyl form of the CCRs give rise to an abstract C*-algebra called the Weyl algebra. All of these abstract algebras are equivalent up to *-isomorphisms.

While Ruetsche[113] claims that the algebraic imperialist is a fictionalized extreme position, she clearly has non-fictional people in mind such as Robinson, Segal, Haag, and Kastler.[114] To the extent that algebraic imperialism is supposed to be a reflection of their views, Ruetsche's claim that the possible states are all elements of $\mathcal{A}_1^{*+}$ is incorrect. In fact, they have stated that the "physical" states are a subset of $\mathcal{A}_1^{*+}$. For example, Haag (1996, 128) suggests that mathematical states which have infinite energy are not physically realizable states and should be excluded from the physical state space. Segal (1992, 145) gives an example of states that have infinite expectation values for functions of position and momentum. Such states are not empirically accessible or observable. Physical states for Segal (1961, 7) (1967, 120 and 132) are regular states on the Weyl algebra.[115] Given the algebraic imperialist's penchant for considering only local observables as physical (since they can be locally measured), it makes sense that they may not want to think of every mathematical

113 Ruetsche (2007) has introduced a distinction between *apologetic imperialism* and *bold imperialism* within algebraic imperialism. The type of algebraic imperialist that has been discussed so far she classifies as an apologetic imperialist. Among other things, the bold imperialist regards the norm topology as being physically significant while the weak topology is not. The bold imperialist position will not be discussed in what follows.

114 See their quotes in section 4.5.

115 Roughly, a state on a Weyl algebra over a symplectic space is said to be *regular* if its GNS representation is such that for the generators of the algebra it is strongly continuous (for more information, see (Arageorgis, Earman, and Ruetsche 2002a, 191)).

state as a physically realizable state. Thus, on operationalist grounds the state space should be restricted to the set of physically realizable states $\mathcal{A}_{1P}^{*+} \subset \mathcal{A}_{1}^{*+}$.[116]

Segal (Baez, Segal, and Zhou 1992, 145) later argued that not all elements of an algebra should be considered observables. For example, when doing quantum mechanics on a line the observable $\cos p + \frac{1}{1+q^2}$ (formed from the position $q$ and momentum $p$ observables) is a bounded linear hermitian operator, but it does not correspond to any measurable physical quantity or possible experiment known. Thus, the physical observables might be a proper subset $\mathcal{A}_P \subset \mathcal{A}$ of the C*-algebra $\mathcal{A}$. Of course, one can always restrict the physical content of any of the mathematical options in the previous diagram to a "physical" subset according to some criteria of what counts as "physical." There are at least four possible types of algebraic imperialist depending on whether they are physically liberal or physically conservative with regards to observables and states.[117]

[116] The state space could be restricted in other ways. For example, states could be required to be regular, satisfy the Hadamard condition, and / or satisfy the KMS condition. Section 5.3 discusses some of these possibilities in more detail.
[117] The liberal / conservative terminology is inspired by (Clifton and Halvorson 2001), but my use is different from theirs.

| | Liberal about States | Conservative about States |
|---|---|---|
| Liberal about Observables | $\left(\mathcal{A},\mathcal{A}_1^{*+}\right)$ | $\left(\mathcal{A},\mathcal{A}_{1P}^{*+}\right)$ |
| Conservative about Observables | $\left(\mathcal{A}_P,\mathcal{A}_1^{*+}\right)$ | $\left(\mathcal{A}_P,\mathcal{A}_{1P}^{*+}\right)$ |

Table 5.1: Types of Algebraic Imperialism

There is also some ambiguity about what kind of algebra the imperialist will choose. For example, in (Ruetsche 2002, 316) the algebraic imperialist will choose the Weyl C*-algebra $\mathcal{W}$, while in later papers (especially, (2007)) the algebra is just a C*-algebra of self-adjoint operators. But there are many other possible choices for the imperialist. They could choose the Weyl C*-algebra for the canonical commutation relations (CCRs) or the canonical anticommutation relations (CARs). It could be a C*-algebra of self-adjoint operators or essentially self-adjoint operators or bounded operators. Some researchers such as Haag thought that the algebras $\mathcal{A}(O)$ defined on bounded regions of spacetime $O$ were the truly operationally significant algebras since all measurements are local. If the imperialist did not want to privilege a particular $O$, she could choose the quasi-local algebra $\mathcal{A}_{\text{loc}}=\overline{\bigcup_{O\in\mathcal{M}}\mathcal{A}(O)}$.[118] The Weyl algebra (CCRs or CARs) or

[118] An algebraic imperialist in AQSM would choose the C*-algebra $\mathcal{A}(s,t)$ for a bounded region of space $s\in\mathbb{R}^3$ ($\mathbb{R}^3$ is Euclidean space) at time $t\in\mathbb{R}$, or the quasi-local algebra $\mathcal{A}_{\text{loc}}(t)=\overline{\bigcup_{s\in\mathbb{R}^3}\mathcal{A}(s,t)}$ at $t$, or even the quasi-local algebra $\mathcal{A}_{\text{loc}}=\overline{\bigcup_{t\in\mathbb{R}}\mathcal{A}_{\text{loc}}(t)}$ for all $t$.

C*-algebras of bounded or self-adjoint operators can also all be defined on different spacetime regions $O$ (e.g., the Rindler wedges or double diamonds) or as a quasi-local algebra $\mathcal{A}_{\text{loc}}$. There are other abstract algebras the imperialist can choose: the W*-algebra $\mathcal{A}^{**}$, Jordan algebras, Segal algebras, *-algebras, or Clifford algebras. Further, the imperialist can be liberal or conservative about each of these algebras and the set of states defined on them. The algebraic imperialist has a much broader range of options than the ones Ruetsche provides.

Ruetsche's argument against algebraic imperialism has changed over these papers. In her (2002) and (2007) papers, she uses Summers' (2001) argument, which was discussed in section 4.5, against HK-physical equivalence.[119] However, in her (2003) and (2006) papers, she criticizes the algebraic imperialist position on the grounds that it cannot account for the differences between UIRs for two reasons. (1) Based in part on the argument in (Kronz and Lupher 2005), the temperature observable is not an element of $\mathcal{A}$. So, the algebraic imperialist recognizes no observable that accounts for the differences in states based on temperature. Also, the observables that belong to the algebra of observables at infinity $\mathfrak{B}_{\pi_\varphi}^{\infty}$ are not elements of $\mathcal{A}$. (2) Nor can the algebraic imperialist account for phase transitions or symmetry breaking which require different UIRs.

[119] More specifically, her (2007) paper uses Summers' argument against the apologetic imperialist.

As discussed at the end of section 4.5, her arguments do not show in these cases that the mathematical condition for HK-physical equivalence is violated. A hard-core C*-algebraic imperialist could refuse to acknowledge these different UIRs as having any significant physical differences if the condition of HK-physical equivalence is satisfied. But the arguments given in sections 4.6-4.7 show that the mathematical condition for HK-physical equivalence will not be satisfied for most of the representations that are used in physical examples. In fact, when the representation is irreducible, is generated from a KMS state, or is a type III factor representation the mere unitary inequivalence of two C*-representations is sufficient to show that their W*-representations are not HK-physically equivalent. These arguments apply to the C*-algebraic imperialist regardless of whether the C*-algebra is the Weyl algebra (CARs or CCRs) or a C*-algebra of bounded or self-adjoint operators.

### 5.3 HILBERT SPACE CONSERVATIVE

The other "extreme" interpreter of QFT is the Hilbert space conservative. Hilbert space conservatives identify physically relevant observables with the set of bounded self-adjoint operators $\mathcal{B}_{sa}\left(\mathcal{H}_{\pi_\omega}\right)$ on some particular Hilbert space $\mathcal{H}_{\pi_\omega}$ [120] and the physically possible states with the set of density operators $\mathfrak{T}_{\pi_\omega}$

[120] Sometimes (2006) she assumes that the Hilbert space $\mathcal{H}_{\pi_\omega}$ is separable, but non-separable Hilbert spaces can also be considered.

defined on $\mathcal{H}_{\pi_\omega}$.[121] The Hilbert space conservative can rephrase the physical content of QFT using the algebraic language. In her latest paper (2007), the physical content of QFT for the Hilbert space conservative is ($\pi_\omega(\mathcal{A})''$, $\mathfrak{F}_\omega$). If the abstract state $\omega$ is a pure state, this implies that the representation $\pi_\omega(\mathcal{A})$ is irreducible and therefore isomorphic to $\mathcal{B}_{sa}\left(\mathcal{H}_{\pi_\omega}\right)$. By the reverse GNS theorem (see section 4.2 above), all density operators $\mathfrak{F}_{\pi_\omega}$ acting on $\mathcal{B}_{sa}\left(\mathcal{H}_{\pi_\omega}\right)$ will have abstract counterparts in the folium $\mathfrak{F}_\omega$ of $\pi_\omega(\mathcal{A})$. Thus, according to Ruetsche the algebraically-minded (???) Hilbert space conservative will assert that the physical content of QFT, which originally was ($\mathcal{B}_{sa}\left(\mathcal{H}_{\pi_\omega}\right)$, $\mathfrak{F}_{\pi_\omega}$), should be repackaged as: ($\pi_\omega(\mathcal{A})''$, $\mathfrak{F}_\omega$).

Like the algebraic imperialist, the Hilbert space conservative has a number of other possible choices for filling out the physical content of (O, S). With respect to observables, the observables could be bounded operators $\mathcal{B}\left(\mathcal{H}_{\pi_\omega}\right)$ on a Hilbert space, self-adjoint operators $\mathcal{B}_{sa}\left(\mathcal{H}_{\pi_\omega}\right)$, or essentially self-adjoint operators $\mathcal{B}_{esa}\left(\mathcal{H}_{\pi_\omega}\right)$. An algebraically-minded Hilbert space conservative could also associate their Hilbert space with a C*-representation $\pi_\omega(\mathcal{A})$ or a von Neumann algebra $\pi_\omega(\mathcal{A})''$. The algebraic state $\omega$ used to construct the

[121] In her (2002) paper, the Hilbert space that the conservative selects is a representation of the

representation need not be a pure state as Ruetsche suggested above. Ruetsche (2002, 359) says that the Hilbert space need not be some simple, non-composite Hilbert space. In cases like this, the Hilbert space can be a direct sum or tensor product, so multiple abstract states can be combined (e.g., $\varphi = \sum_{i=1}^{n} \lambda_i \psi_i$ of abstract states $\psi_i$, where $\lambda_i > 0$, $\varphi \neq \psi_i$, and $\sum_{i=1}^{n} \lambda_i = 1$) and a representation $\pi_\varphi$ can be constructed via the GNS theorem.

Ruetsche has two different arguments to show the limitations of the Hilbert space conservative. (1) The only physically possible states for the Hilbert space conservative are the states belonging to the folium $\mathfrak{F}_\varphi$ of $(\mathcal{H}_{\pi_\varphi}, \pi_\varphi(\mathcal{A}))$. The Hilbert space conservative does not recognize the physical significance of UIRs just as the algebraic imperialist does not. If $\pi_\psi(\mathcal{A})$ is a UIR with respect to $\pi_\varphi(\mathcal{A})$, then their folia have no states in common.[122] The states belonging to a UIR are not physically possible for the Hilbert space conservative. However, there are not enough states in $\pi_\varphi(\mathcal{A})$ to model different temperatures (2002) (2003) (2006) (2007) , phase transitions (2003) (2006) (2007), or ergodicity (2006) in AQSM in the thermodynamic limit.[123] For example, states belonging to the folia of two representations associated with KMS states at different temperatures will not both be physically possible for the Hilbert space

Weyl relations, but this requirement is not used in her subsequent papers.

[122] More specifically, if they are disjoint, then by condition (3) of theorem 4.7.2.1 the intersection of their folia is the empty set.

conservative. Thus, non-actual temperatures would be physically impossible which seems "modally draconian" (2002, 364). These arguments are effective against the Hilbert space conservative in the domain of AQSM, but they may not bother the conservative in AQFT. There may be a principle in AQFT that allows the Hilbert space conservative to uniquely select one Hilbert space.

Ruetsche's (2002) (2007) second argument against the Hilbert space conservative in the context of AQFT focuses on what principle the conservative could use to select a particular Hilbert space as the repository of physical content which relegates the Hilbert spaces associated with all other UIRs to being mathematical artifacts. One candidate for such a principle is the Hadamard condition.[124] If the universe is closed, then it admits compact Cauchy surfaces and all Hadamard vacuum states define unitarily *equivalent* representations. Thus, no UIRs would exist if only abstract states which satisfy the Hadamard condition were used to construct representations. However, for closed universes the evolution of a Hadamard state need not be to another Hadamard state, so UIRs might be possible at different times. For an open universe all Hadamard states would not give rise to unitarily *equivalent* representations. Further, neither the Boulware vacua of extended Schwarschild spacetime nor the Unruh vacuum satisfy the Hadamard condition. In Kerr spacetime, there are no states that

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[123] This assumes that the representation is a factor or irreducible.

[124] Roughly, if a state has an expectation value for its two point function of the field which exhibits a prescribed singularity structure as two spacetime points approach one another, then it satisfies the Hadamard condition. Satisfying the Hadamard condition supports the expectation value assignment of the stress-energy tensor. For more information, see (Wald 1994).

satisfy the Hadamard condition. Thus, the Hadamard condition will only provide the Hilbert space conservative with a unique Hilbert space if the universe is closed, but even then the evolution of Hadamard states need not evolve into other Hadamard states.

Another possible principle might involve some symmetry of spacetime such as having global timelike isometries. Given a number of conditions such as the vacuum state being invariant under the full isometry group of the spacetime, there is a unique representation that has a natural particle-interpretation for Minkowski spacetime. However, spacetimes which do not satisfy these conditions will not specify a single class of unitarily equivalent representations. For example in QFT on curved spacetime, there is no reason to expect that a generic spacetime will share the symmetries of Minkowski spacetime.

The better response to the Hilbert space conservative in the context of QFT was given in the third chapter in response to the *Fock-representation chauvinist*. In the context of canonical QFT, the Fock-representation chauvinist is a Hilbert space conservative that privileges the Fock space constructed using the bare no-particle vacuum state and the *a*-operators. Support for this position comes from a theorem (Putnam 1967, 86) which states that if there exists a vacuum state that is annihilated by the annihilation operator, then the Fock representation is unique. However, for systems with an infinite number of degrees of freedom, there are many different vacua possible. Further, singling out one representation which has a translation invariant vacuum falls afoul of

Haag's theorem. Thus, the Fock-representation chauvinist as well as the Hilbert space conservative must deal with the consequences of Haag's theorem, namely, that they will not be able to model interactions. If the Hilbert space conservative finds some principle that singles out one class of unitarily equivalent representations, she must also show that this Hilbert space is capable of modeling all possible interactions, which is extremely unlikely.[125]

Of course, a *Hilbert space liberal* could choose a Hilbert space that would capture more of the physical content of QFT. The Hilbert space liberal could choose the non-separable Hilbert space created by taking an infinite direct product of all of the unitarily inequivalent Fock spaces. This van Hove-like "universal receptacle" would include all of the cases that the Hilbert space conservative would classify as physically impossible such as Fock spaces with different masses or coupling constants. In the context of canonical QFT, this universal receptacle would be the best mathematical structure for capturing the physical content of UIRs.[126] This is in the spirit of the approach adopted here for AQFT. In the case of the Unruh effect in AQFT, the direct sum of the Hilbert spaces associated with the Rindler representation and the Minkowski representations with different temperatures in the Rindler wedges could be used.

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[125] There are ways around this which include putting the system in a box and then taking the limit as the length of the box goes to infinity or imposing a momentum cut-off and then removing the cut-off. However, there is no reason to think that the resulting Hilbert space or Fock space, if it exists, will be unitarily equivalent to the old Hilbert space of Fock space. In fact, the renormalized Fock space is often not unitarily equivalent to the Fock representation (see (Glimm 1969)).

## 5.4 MIXED APPROACHES

Based on Clifton and Halvorson's (2001) distinction between a liberal or conservative with respect to states or observables, Ruetsche (2007) discusses "mixed approaches" that combine elements of the algebraic imperialist and the Hilbert space conservative. These are merely abstract possibilities that no one has actually defended as viable positions on physical content in AQFT. For example, one can be an algebraic imperialist with respect to observables and an algebraically-minded Hilbert space conservative with respect to states, so that the physical content is: ($\mathcal{A}$, $\mathfrak{F}_\omega$). Such an approach would not recognize enough observables (e.g., temperature) nor enough states (e.g., states that have different temperatures). One could also be a Hilbert space conservative with respect to observables and an algebraic imperialist with respect to states, in which case the physical content is: ($\pi_\omega(\mathcal{A})''$, $\mathcal{A}_1^{*+}$). This approach also would not have a temperature observable. In addition, it might recognize some states as being physically meaningful which are not physically realizable, such as states with infinite energy. Thus, these mixed approaches do not overcome any of the difficulties that afflict the algebraic imperialist and Hilbert space conservative since they do not recognize enough states or enough observables.

[126] However, there may be problems using this universal receptacle to capture interactions (see (Wightman 1967)).

## 5.5 HILBERT SPACE UNIVERSALIST

An alternative to the position of the algebraic imperialist and the Hilbert space conservative which invests physical significance in UIRs was sketched in (Kronz and Lupher 2005). The idea is that the universal enveloping von Neumann algebra $\pi_u(\mathcal{A})''$ is a more appropriate structure for capturing the physical content of AQFT and AQSM than $\mathcal{A}$. Observables such as temperature and chemical potential would not be excluded as they are for the algebraic imperialist and the Hilbert space conservative. The norm topology is too strict for the construction of such observables whereas the weak topology allows for new observables to used.[127] This position would also have enough states to account for phenomena such as phase transitions. Thus, it has enough states and observables to overcome the criticisms of the algebraic imperialist and the Hilbert space conservative. Ruetsche (2007) calls our position universalism, though I prefer *Hilbert space universalist*. While Ruetsche offers no criticism of the position in her (2003) paper, her current work (2006) (2007) offers several criticisms of our position. These criticisms are supposedly overcome by her position which I call *Hilbert space pluralism*. Before examining these arguments, it will helpful to discuss the development of her position.

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[127] One could believe like Segal that the weak topology has no intrinsic significance and be a C*-Hilbert space universalist, in which case the physically significant observables would belong to $\pi_u(\mathcal{A})$. However, this would entail that observables such as temperature and chemical potential are not physically significant.

## 5.6 HILBERT SPACE PLURALIST

Ruetsche wants to navigate an approach to physical possibilities that is more moderate (less restrictive) than either the algebraic imperialist or the Hilbert space conservative. This approach takes UIRs seriously as having physical content. Though the name she gives her position changes with each paper, I call it *Hilbert space pluralism*. Her first proposal (2002, 376-377) selects the state space of the algebraic imperialist. The observables in the C*-algebra $\mathcal{A}$ form a core constituency of observables which are supplemented by observables created at the concrete level by closing a particular representation in the weak operator topology. Since the algebraic imperialist in this paper is using the Weyl C*-algebra $\mathcal{W}$, the physical content is $\left(\mathcal{W}, \mathcal{W}_1^{*+}\right)$. She classifies these new observables as state-dependent observables.[128] While the observables in $\mathcal{W}$ are supposed to characterize physical possibility in a broad sense, these state-dependent observables characterize physical possibility in a narrow sense. Dynamical considerations, such as imposing the Hadamard condition, could serve to restrict the state space to a proper subset of $\mathcal{W}_1^{*+}$.

In her (2003, 1339-1342) paper, Ruetsche proposes what she calls a "Swiss army approach" to specifying the physical content of a physical theory that is based on an idea she attributes to Kadison. The specification of physical

[128] As the last chapter discussed in detail, these observables do not have abstract counterparts in $\mathcal{W}$ or more generally in $\mathcal{A}$, but they do have abstract counterparts in $\mathcal{W}^{**}$ and $\mathcal{A}^{**}$.

content has two tiers. The first tier selects the broadest set of possibilities that are *a priori* in that no physical contingencies are taken into account.

Tier I: $\left(\mathcal{W}_{\text{sa}}, \mathcal{W}_1^{*+}\right)$

The first tier takes the self-adjoint elements of the Weyl algebra $\mathcal{W}_{\text{sa}}$ and its abstract states as the broadest set of physical possibilities. At the second tier, physical contingencies are taken into account. A smaller subset of states $\mathcal{P} \subset \mathcal{W}_1^{*+}$ relevant to the physical situation are chosen from $\mathcal{W}_1^{*+}$. The GNS representations of those states are formed and then closed in the weak operator topology. These new observables, observables "parochial" to the particular representation, are added to the observables in $\mathcal{W}_{\text{sa}}$.

Tier II: $\left(\left(\pi_{\omega_1}\left(\mathcal{W}_{sa}\right)'', \pi_{\omega_2}\left(\mathcal{W}_{sa}\right)'', \ldots\right), \omega_1, \omega_2, \ldots \in \mathcal{P}\right)$

Ruetsche also considers how this Swiss army approach could be implemented using the universal representation. Here the idea is that the universal representation is like a Swiss army knife with its blades folded up.

Tier I: $\left(\pi_u\left(\mathcal{W}_{\text{sa}}\right), \mathcal{W}_1^{*+}\right)$

The next (coalescence) stage appeals to contingent features of the physical situation to focus on a small subset of representations in the universal representation.

Tier II: $\left(\left(\pi_{\omega_1}(\mathcal{W}_{sa})'', \pi_{\omega_2}(\mathcal{W}_{sa})'', \ldots\right), \omega_1, \omega_2, \ldots \in \mathcal{P}\right)$

At this stage, we know which blades in the Swiss army knife are useful for the physical situation at hand and the von Neumann algebras are drafted to serve to explain various sorts of phenomena such as symmetry breaking and the construction of a temperature observable.

In her (2006) paper, she keeps the two tier model, but she changes the abstract algebra from the Weyl algebra to the self-adjoint portion of an abstract C*-algebra $\mathcal{A}_{sa}$.

Tier I: $\left(\mathcal{A}_{sa}, \mathcal{A}_1^{*+}\right)$

The second tier uses physical contingencies to select a subset $\Omega_c \subset \mathcal{A}_1^{*+}$ of states and uses them to build to build GNS representations. The selection of $\Omega_c$ is done from pragmatic considerations *P*. Though it is not mentioned specifically in the discussion of her position, she clearly wants to consider the von Neumann extensions of these GNS representations and not just the C*-representations as coalescing the content.

Tier II: $\left(\left(\pi_{\omega_1}(\mathcal{A}_{sa})'', \pi_{\omega_2}(\mathcal{A}_{sa})'', \ldots\right), \omega_1, \omega_2, \ldots \in \Omega_c \subset \mathcal{A}_1^{*+}\right)$

On the coalescence account developed in this paper, the content of a quantum theory is a pair $\left(\left(\mathcal{A}_{\mathrm{sa}}, \mathcal{A}_1^{*+}\right), f\colon P \to \left(\mathcal{C}, \Omega_{\mathrm{c}}\right)\right)$, where the first pair $\left(\mathcal{A}_{\mathrm{sa}}, \mathcal{A}_1^{*+}\right)$ is the first tier and the second tier involves a function *f* that uses pragmatic considerations *P* to select the observables $\mathcal{C}$ from each $\pi_{\omega_1}(\mathcal{A}_{\mathrm{sa}})'', \pi_{\omega_2}(\mathcal{A}_{\mathrm{sa}})'', \ldots$ where each state comes from certain selected states $\omega_1, \omega_2, \ldots \in \Omega_c$. What the algebraic imperialist got right was capturing the broadest sorts of possible physical content in quantum theory. The Hilbert space conservative is correct that a concrete Hilbert space coalesces content. Both positions miss out on this second stage of content; the algebraic imperialist suppresses it entirely while the Hilbert space conservative privileges a single Hilbert space for the content of the whole theory.

In her (2007) paper, Ruetsche sketches a position she calls *tempered universalism* that is supposed to be less extravagant than Hilbert space universalism but more generous than either algebraic imperialism or the Hilbert space conservative. On this approach, there is criterion that selects a "physically reasonable" subset of algebraic states from $\mathcal{A}_1^{*+}$. For each state in this subset, its von Neumann algebra is constructed. All of these von Neumann algebras will then be direct summed together. Depending on the situation at hand, different principles, such as states satisfying the Hadamard condition, can be used to select the appropriate states and hence the observables defined by the direct sum of their von Neumann algebras.

In summary, the Hilbert space pluralist specification of physical content is done on a case by case basis; it is not formally systematic. It starts with the same framework of $\left(\mathcal{A}_{sa}, \mathcal{A}_1^{*+}\right)$ and then attempts to refine it. However, this is more of a shift in emphasis than in content ultimately. Ruetsche wants to use these new observables in specific applications, but she does not acknowledge where she is getting these new observables, namely, $\mathcal{A}^{**}$.

## 5.7 RUETSCHE'S CRITICISMS OF THE HILBERT SPACE UNIVERSALIST

Though Ruetsche formulates a version of her Swiss army approach using the universal enveloping von Neumann algebra, her later papers have criticized this position. Her main criticism (2006) (2007) is that Hilbert space universalism lacks "interpretive good taste." The "parochial" observables are observables that are created in a particular representation by closing the representation in the weak operator topology. These observables do not have abstract counterparts in $\mathcal{A}$. Ruetsche claims that *most* of these parochial observables fail the main function of observables which is to discriminate between different physical situations. Suppose *B* is a parochial observable for $\pi_{\varphi}(\mathcal{A})''$. If $\pi_{\psi}(\mathcal{A})$ is disjoint from $\pi_{\varphi}(\mathcal{A})$, then every state in the folium $\mathfrak{F}_{\psi}$ has an expectation value of zero for *B*. However, Ruetsche thinks that the temperature observable, which is a parochial observable without an abstract counterpart in $\mathcal{A}$, does discriminate between physical situations since it takes different values for states in different folia.

If an observable is to be dismissed as mathematical surplus structure on the grounds that the expectation values of the states in its folium are zero for an observable, then many observables in non-relativistic quantum mechanics fail this criterion.[129] For example, consider the eigenstates for spin in the *x*-direction $S_x$. The expectation values of *all* eigenstates of $S_x$ are *zero* for spin in the *y*-direction $S_y$ and spin in the *z*-direction $S_z$. Thus, on Ruetsche's criteria $S_y$ and $S_z$ would be "irrelevant" for eigenstates of $S_x$. $S_y$ and $S_z$ would be considered mathematical surplus structure which is clearly incorrect since they are physical observables that can be measured in a laboratory.

Further, the Hilbert space universalist never claimed that *every* observable in $\pi_u(\mathcal{A})''$ has physical significance. It was only claimed that it provides us with a large set of new observables that do not have abstract counterparts in $\mathcal{A}$ which may be useful in both AQFT and AQSM. For specific physical situations, a

[129] I would like to thank David Baker and Hans Halvorson for a discussion which illuminated the criticism developed in this paragraph.

subset of $\pi_u(\mathcal{A})''$ may be sufficient to model all of the physical aspects one is interested in. Does Hilbert space pluralism really avoid surplus structure and capture all of the physical content left out by the algebraic imperialist and the Hilbert space conservative?

### 5.8 Criticisms of Hilbert Space Pluralism

Hilbert space pluralism does not avoid surplus structure any more than the Hilbert space universalist, algebraic imperialist, or even an interpreter of ordinary non-relativistic quantum mechanics. The picture Ruetsche presents of how to construct von Neumann algebras is the following. Start with a C*-algebra. Choose an abstract state and build its GNS representation. Finally, close that C*-representation in the weak operator topology. The diagram for this procedure does not use W*-representations.

$$\begin{tikzcd}
 & \underline{\text{C*-Algebra}} & & \underline{\text{W*-Algebra}} \\
\text{Abstract:} & \mathcal{A} \arrow[r, "*"] \arrow[ddd, "\pi_\omega"'] & \mathcal{A}^* & \\
 & & \cup & \\
 & & \omega \in \mathcal{A}_1^{*+} & \\
\text{Concrete:} & \pi_\omega(\mathcal{A}) \arrow[rr, "'' \text{ or } -"] & & \pi_\omega(\mathcal{A})'' = \pi_\omega(\mathcal{A})^-
\end{tikzcd}$$

Figure 5.1: Algebraic Structures for the Hilbert space Pluralist

This is the standard account of how to construct von Neumann algebras. However, as the full diagram at the beginning of this chapter shows, von Neumann algebras can also be constructed using W*-representations of the bidual $\mathcal{A}^{**}$ and $\pi_\omega(\mathcal{A})'' = \pi_{\tilde{\omega}}^{w}(\mathcal{A}^{**}) = \pi_\omega(\mathcal{A})^-$. As discussed in the last chapter, the universal enveloping von Neumann algebra $\pi_u(\mathcal{A})''$ is isomorphic to $\mathcal{A}^{**}$. All of the "parochial" observables in $\pi_u(\mathcal{A})''$ have abstract counterparts in $\mathcal{A}^{**}$. Thus, each von Neumann algebra in the direct sum of von Neumann algebras that the Hilbert space pluralist uses is a W*-representation of *all* of the abstract counterparts of *every* element in $\pi_u(\mathcal{A})''$. It might be that a large number of the elements in $\mathcal{A}^{**}$ are mapped to the identity operator or zero operator in the von

Neumann algebra's Hilbert space, but this does not change the fact that every von Neumann algebra is a representation of all the elements in $\mathcal{A}^{**}$.

Further, there is no guarantee that every parochial observable in every von Neumann algebra in the Hilbert space pluralist's direct sum is a physical observable, much less that every non-trivial element in the von Neumann algebra is a physical observable or physically relevant for the particular situation. Depending on the abstract C*-algebra selected, there may be surplus structure or irrelevant elements that are represented as non-trivial elements in *each* von Neumann algebra in the direct sum. Ruetsche wants a temperature observable, but the construction described in the last chapter uses a continuum of von Neumann algebras.[130] While the set of abstract states used to construct the temperature observable is a subset of the set of all abstract states, how much surplus structure is really being avoided if a continuum of von Neumann algebras is being used to construct the temperature observable?

If Ruetsche wants to avoid surplus structure at the concrete level by restricting which abstract states are going to be selected, then should there also not be a restriction on which elements of the abstract algebra are physical and which are merely mathematical possibilities? Different abstract algebras can be selected and different subsets of those algebras might be relevant for the physical situation at hand. Ruetsche focuses on the self-adjoint portion of the Weyl algebra, but there might also be bounded operators in the algebra that can be called into service. The physical situation might involve fermions, in which case the C*-algebra of the Weyl Relations of the CARs would be the appropriate abstract algebra instead of the C*-algebra of the Weyl relations of the CCRs. It is also likely that not all of the observables belonging to the Weyl algebra of the CCRs or CARs will be a physically measurable observable. Linear combinations or products of observables will be elements of the algebra, but are we to take all of these elements seriously as physical observables? Consider the operator $B=\sum_{i=1}^{n} A_i$ where $A_i \in \mathcal{A}$ and $\mathcal{A}$ is a C*-algebra. If $B$ is supposed to be a physical observable, does this imply that every $A_i$ is also a physical observable? Would the operator $B'=\sum_{i=1}^{n}\alpha_i A_i$ ($\alpha_i \in \mathbb{C}$) also count as a physical observable? These observables are elements of the abstract algebra and would have a nontrivial representation if at least one $A_i$ has a nontrivial representation in a particular von Neumann algebra. The Weyl algebra only contains elements built up from position and momentum, but surely we also want spin observables. However, these are not elements of the Weyl algebra. Thus, if the goal of the Hilbert space pluralist is to remove as much surplus structure as possible, then this must be applied not only at the level of parochial observables and abstract states, but also for the abstract algebra. Further, each von Neumann algebra has concrete versions of every abstract element in the C*-algebra. If the Hilbert space pluralist

130 Further, an abstract temperature observable can be constructed in $\mathcal{A}^{**}$ without using the von Neumann algebras as discussed in the previous chapter.

really wants to avoid surplus structure, then she would need to articulate why *all* of these copies of $A \in \mathcal{A}$ in each von Neumann algebra $\pi_{\omega}(A)''$, $\pi_{\varphi}(A)''$, … are physically necessary. The bottom line is that the Hilbert space pluralist must try to minimize surplus structure while making sure that she has enough observables and states to capture all of the relevant features of the physical situation, but Ruetsche has given us no criteria by which to do this.

The Hilbert space pluralist wants to pick certain elements of $\mathcal{A}^{**}$ and use them in certain situations but not acknowledge where they come from. To use an analogy, $\mathcal{A}^{**}$ is like a big warehouse that has a number of boxes in it.[131] The Hilbert space pluralist goes into the warehouse and grabs a few boxes. When people ask the Hilbert space pluralist where he got the boxes, he says that you can order them from a small convenience store around the corner. Of course, the small convenience store will order them from the warehouse!

## 5.9 BIDUALISM

A typical interpreter of non-relativistic quantum mechanics will identify the observables with the set of all self-adjoint operators defined on a Hilbert space and the states as the collection of all density operators defined on a Hilbert space. If you push this interpreter on whether *every* self-adjoint operator and *every* density operator are real physical quantities, they would deny such a

[131] I would like to thank Fred Kronz for this analogy.

strong claim. The pragmatic answer would be that the physical observables and states are some proper subset, but that using these larger sets is conceptually helpful, i.e., one can prove theorems about self-adjoint operators and density operators and those theorems will apply to the physical observables and states. These larger collections of observables and states are sufficient for modeling many of the physical situations encountered in ordinary quantum mechanics. It is in this spirit that the position of *bidualism* is offered as an account of the physical content of AQFT and AQSM.[132]

For the bidualist, the observables are elements of $\mathcal{A}^{**}$. The states are the normal states defined on $\mathcal{A}^{**}$ and will be denoted as $\mathcal{N}$. Recall that each abstract state $\phi$ in $\mathcal{A}_1^{*+}$ has a unique extension $\tilde{\phi}$ to a normal state on $\mathcal{A}^{**}$. Using this normal state, the GNS representation of $\mathcal{A}^{**}$ is a von Neumann algebra that is equivalent to constructing the GNS representation from $\phi$ and then taking the bicommutant: $\pi_{\phi}(\mathcal{A})'' = \pi_{\tilde{\phi}}\left(\mathcal{A}^{**}\right)$. Thus, the physical content of AQFT and AQSM for the bidualist is : ($\mathcal{A}^{**}, \mathcal{N}$).

These mathematical structures overcome the disadvantages of both the algebraic imperialist and Hilbert space conservative. $\mathcal{A}^{**}$ contains the abstract counterparts to the "parochial" observables as well as all of the observables in the original C*-algebra $\mathcal{A}$. It contains the building blocks for observables such as temperature and chemical potential, which are not elements of $\mathcal{A}$, and

[132] I am indebted to Hans Halvorson for the suggestion that the position developed here be called

enough states to model states with different temperatures or different phases. It also provides a larger collection of classical observables than $\mathcal{A}$. Based on theorem 4.8.2, $\mathcal{A}^{**}$ is the mathematical structure in which the differences between different UIRs can be described.

However, bidualism should *not* be understood as claiming that *every* element in $\mathcal{A}^{**}$ is an observable nor that *every* state in $\mathcal{N}$ corresponds to a physical state. For specific situations, perhaps only one state and its associated von Neumann algebra will be necessary or relevant. However, if the situation requires multiple states or an observable constructed from a direct sum, then that can be constructed as well. This is similar to the interpreter of quantum mechanics. Though the physical states are the collection of all density operators defined on a Hilbert space, in many situations only one particular density operator and a few self-adjoint observables will be necessary to model a particular situation.

What is the connection between the Hilbert space Universalist and the Bidualist? Since $\pi_u(\mathcal{A})''$ and $\mathcal{A}^{**}$ are isomorphic, they both recognize all of the same observables. It is not difficult to see that the state spaces are also isomorphic since each normal state in $\mathcal{N}$ is the unique extension of a state in $\mathcal{A}_1^{*+}$. Thus, they both specify the same physical content but one does so at an abstract level while the other does so at the concrete level. *The distinction between the algebraic imperialist and the Hilbert space conservative collapses*

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bidualism and to Fred Kronz for the "spirit" of the position..

*once a large enough abstract algebra and Hilbert space are chosen*. Bidualism is a kind of algebraic imperialism which uses a sufficiently large algebra to capture the differences between different UIRs. The bidualist is also free to specify which abstract algebra to use. For example, the bidual of the Weyl algebra $\mathcal{W}^{**}$ of the CCRs or CARs could be chosen or the bidual of the quasi-local C*-algebra $\mathcal{A}_{\text{loc}}$. The bidualist might decide that they want a more general collection of observables. By taking the tensor product of multiple C*-algebras (e.g., $\mathcal{W}_{CCRs} \otimes \mathcal{W}_{CARs} \otimes \mathcal{A}_{\text{loc}}$) a tensor product of W*-algebras can be defined (very roughly, the W*-algebra tensor product is defined in terms of $\left(\mathcal{W}_{CCRs} \otimes \mathcal{W}_{CARs} \otimes \mathcal{A}_{\text{loc}}\right)^{**}$; for more information see section 4.3 of (Bing-Ren 1992)).

However, being a bidualist allows for a certain amount of flexibility over the Hilbert space universalist in terms of creating new observables. For example, the form of any observable in the universal representation is $\pi_u(A) = \pi_\phi(A) \oplus \pi_\psi(A) \oplus \pi_\omega(A) \oplus \ldots$. However, one might want to create an observable such as $\pi_\phi(A)'' \oplus \pi_\psi(B)'' \oplus \pi_\omega(C)'' \oplus \ldots$, which is technically not in $\pi_u(\mathcal{A})''$ though each individual element $\pi_\phi(A)'', \pi_\psi(B)'', \pi_\omega(C)'',\ldots$ does belong to $\pi_u(\mathcal{A})''$. For example, the total number operator for a representation, which is made up of projectors in its spectral resolution, is affiliated with its von Neumann algebras. The total number operator is not an element of $\mathcal{A}$.

However, as we have seen the total number operator might not exist for a given representation. But number operators for fixed wavefunctions can be constructed, so the probability that a state has a finite number of particles with that particular wavefunction will have an answer.[133] Suppose we want to know the particle content for the right Rindler wedge. To do this a number operator can be constructed for every kind of wavefunction for each representation, e.g., take the direct sum of each number operator $N_{\pi_{\omega_{\mathcal{M}}^{\triangleleft}}(\mathcal{W}_{\triangleleft})''}$ for the Minkowski representation on the right Rindler wedge for every wavefunction and direct sum this with the number operator for the Rindler representation on the right Rindler wedge $N_{\pi_{\omega_{R}^{\triangleleft}}(\mathcal{W}_{\triangleleft})''}$ for every wavefunction. This would give us a "total particle number operator" $N_{\triangleleft}$ for the right Rindler wedge.

The position of bidualism allows the greatest freedom in choosing both observables and states. A particular physical situation might use only a small subset of these observables and states, but we want a flexible mathematical structure that can model many different types of phenomena in both AQFT and AQSM. Further, we want a mathematical structure that allows us to compare different UIRs and deploy them in specific situations either individually or as a group. Bidualism ($\mathcal{A}^{**}$, $\mathcal{N}$) is mathematically simpler and more conceptually unified than Hilbert space pluralism $\left(\left(\pi_{\omega_1}(\mathcal{A}_{sa})'', \pi_{\omega_2}(\mathcal{A}_{sa})'', \ldots\right), \omega_1, \omega_2, \ldots \in \Omega_c \subset \mathcal{A}_1^{*+}\right)$. If each representation has its

[133] For details of this construction, see section 4.2 of (Clifton and Halvorson 2001).

own "parochial" temperature observable, how should we understand the relationship of the temperature observable associated with one von Neumann algebra with the temperature associated with a disjoint representation? Are these radically incommensurable temperature observables? Does each temperature observable give a temperature of zero for all states in a different UIR? If this is the case, then according to the temperature observable $T_{\pi_{\omega_1}(\mathcal{A}_{sa})''}$ associated with $\pi_{\omega_1}(\mathcal{A}_{sa})''$ the temperature for all states in $\pi_{\omega_2}(\mathcal{A}_{sa})''$ is zero. But according to $T_{\pi_{\omega_2}(\mathcal{A}_{sa})''}$ all the states in $\pi_{\omega_2}(\mathcal{A}_{sa})''$ could have some non-zero temperature! Thus, bidualism is more conceptually unified since there is one temperature observable at the abstract level that each W*-representation has.

Hilbert space pluralism has no claim to having better "interpretive good taste" over bidualism. The "parochial" observables the Hilbert space pluralist wants to use are elements from the bidual warehouse. Further, it is not clear how much surplus structure is really avoided by the Hilbert space pluralist. If a continuum of representations are necessary to build a temperature observable, what advantage does a Hilbert space pluralist have over bidualism? Here is a summary of the main positions with regard to physical content in algebraic approach discussed in this chapter.

Summary of Accounts of Physical Content in the Algebraic Approach

*Algebraic Imperialism*: ( $\mathcal{A}$ , $\mathcal{A}_1^{*+}$ )
*Hilbert Space Conservative*: ( $\pi_\omega(\mathcal{A})''$ , $\mathfrak{F}_\omega$ )
*Mixed Approaches*: ( $\mathcal{A}$ , $\mathfrak{F}_\omega$ ) and ( $\pi_\omega(\mathcal{A})''$ , $\mathcal{A}_1^{*+}$ )
*Hilbert Space Pluralist*: $\left(\left(\pi_{\omega_1}(\mathcal{A}_{sa})'', \pi_{\omega_2}(\mathcal{A}_{sa})'', \ldots\right), \omega_1, \omega_2, \ldots \in \Omega_c \subset \mathcal{A}_1^{*+}\right)$
*Hilbert Space Universalist*: ( $\pi_u(\mathcal{A})''$ , $\mathcal{A}_1^{*+}$ )
*Bidualist*: ( $\mathcal{A}^{**}$ , $\mathcal{N}$ )

Table 5.2: Summary of Different Interpretations of the Algebraic Approach

## 5.10 INCOMMENSURABLE PHYSICAL THEORIES?

It has been argued (Arageorgis 1995) (Arageorgis, Earman, and Ruetsche 2002b) that the particle concepts associated with different UIRs are *incommensurable*. The argument (2002b, 179-180) is based on the fact, which was examined in chapter two, that two unitarily inequivalent Fock spaces (say, our *a*-representation and *b*-representation) will have infinite expectation values, e.g., ${}_b\langle 0|N_a|0\rangle_b = \infty = {}_a\langle 0|N_b|0\rangle_a$. The *a*-representation and *b*-representation would be considered *incommensurable theories*.[134] But incommensurable theories usually have no translation scheme that maps states and observables in theory $T$ to states and observables in theory $T'$.[135] But we do have a translation

[134] Clifton and Halvorson (2001, 457-459) also criticize this argument for claiming that UIRs are incommensurable theories, though they do so from the point of view of the Unruh effect.
[135] I would like to thank Fred Kronz for this suggestion about translation schemes.

scheme in the case of the *a*-representation and *b*-representation! $U_{ab}$ maps any vector $\left|m_k\right\rangle_a$ to $\left|m_k\right\rangle_b = U_{ab}\left|m_k\right\rangle_a = \frac{U_{ab}\left(a^\dagger(\boldsymbol{k})\right)^m \left|0\right\rangle_a}{\sqrt{m!}}$ as well as $a(\boldsymbol{k})$, $a^\dagger(\boldsymbol{k})$ into $b(\boldsymbol{k})$, $b^\dagger(\boldsymbol{k})$. In fact, there is an isomorphism between the sets of vectors and the sets of creation and annihilation operators! When the two representations become unitarily inequivalent $U_{ab}$ ceases to be a *proper* unitary operator, but as an *improper* unitary operator it still shows how to translate say ${}_a\left\langle 0\right|N_a\left|0\right\rangle_a$ into the same experimental question in the *b*-representation, namely, ${}_b\left\langle 0\right|N_b\left|0\right\rangle_b$. It is only when a partial translation occurs that divergent results obtain such as only translating the vacuum state of the *a*-representation for ${}_a\left\langle 0\right|N_a\left|0\right\rangle_a$ into the vacuum state of the *b*-representation ${}_b\left\langle 0\right|N_a\left|0\right\rangle_b = \infty$. Thus, UIRs are not incommensurable physical theories since there is an isomorphism between their states and creation-annihilation operators. The conceptual "paradox" involving UIRs results from only doing a *partial* translation of either the vectors or the total number operator from one UIR to another. Further, there is often a trivial translation scheme for UIRs in the case where the Hilbert or Fock spaces have the same dimension. In those cases, the Hilbert or Fock spaces are isomorphic to each other. It also seems overly dramatic to call say the *b*-representation with coupling constant $g$ and the unitarily inequivalent $b'$-representation with coupling constant $g'$ (where the difference between $g$ and $g'$ can be as small one likes) incommensurable theories. This is similar to the case in Newtonian mechanics where two forces ($F = -k\,x$ and $F' = -k'\,x$) are represented by a spring force but have different spring constants ($k \neq k'$). These two forces are not two fundamentally different forces. Rather, they are two models of the same physical theory.

In the algebraic approach, a different argument could be used to show that different UIRs are incommensurable, namely that disjoint representations have disjoint folia, i.e., if $\pi_\varphi$ and $\pi_\psi$ are disjoint representations of $\mathcal{A}$, then $\mathfrak{F}_\varphi \cap \mathfrak{F}_\psi = \varnothing$. Thus, no density operator in $\mathcal{H}_{\pi_\varphi}$ can be expressed as a density operator in $\mathcal{H}_{\pi_\psi}$ and vice versa. For the extreme Hilbert space conservative, if she selects ($\mathcal{H}_{\pi_\varphi}, \pi_\varphi$), then the physically possible states belong to $\mathfrak{F}_\varphi$ and the states belonging to $\mathfrak{F}_\psi$ are physically impossible. States belonging to disjoint folia assign each other a transition probability of zero. Ruetsche (2007) says that states whose transition probability is zero are *impossible relative to* the states in the folia belonging to a disjoint representation. While it seems that the algebraic approach bolsters the case of those who want to claim that different UIRs are incommensurable theories, the mathematical condition that is the basis for Haag and Kastler's notion of physical equivalence tempers this possibility.

Recall that it is not difficult to find C*-representations that satisfy the mathematical condition of weak equivalence, e.g., faithful representations. Suppose that $\left(\mathcal{H}_{\pi_\varphi}, \pi_\varphi\right)$ and $\left(\mathcal{H}_{\pi_\psi}, \pi_\psi\right)$ are weakly equivalent. It then follows that the closure of the folium $\mathfrak{F}_{\varphi}$ in the weak*-topology on $\mathcal{A}^*$ is equal to the closure of the folium $\mathfrak{F}_{\psi}$ in the weak*-topology on $\mathcal{A}^*$, i.e., ${}^{w^*}\overline{\mathfrak{F}}_{\varphi} = {}^{w^*}\overline{\mathfrak{F}}_{\psi}$. Thus, even if $\pi_{\varphi}$ and $\pi_{\psi}$ are disjoint representations such that $\mathfrak{F}_{\varphi} \cap \mathfrak{F}_{\psi} = \varnothing$, they can each be weakly equivalent to each other in which case ${}^{w^*}\overline{\mathfrak{F}}_{\varphi} = {}^{w^*}\overline{\mathfrak{F}}_{\psi}$. This might seem paradoxical on first glance. To dissolve the "paradox," consider the following analogy. The set of rational numbers $\mathbb{Q}$ is the collection of all numbers of the form $\dfrac{p}{q}$, where *p* and *q* are integers and $q \neq 0$. For any rational number, call it even if $p \cdot q$ is an even number and odd if $p \cdot q$ is an odd number. Based on this property let $\mathbb{Q}_{even}$ be the set of even rational numbers and $\mathbb{Q}_{odd}$ be the set of odd rational numbers. These sets are disjoint in that they have no element in common: $\mathbb{Q}_{even} \cap \mathbb{Q}_{odd} = \varnothing$. However, when they are completed with respect to the metric space (i.e., all the limit points of Cauchy sequences are included) they both are equal to the real numbers, i.e. $\overline{\mathbb{Q}_{even}} = \mathbb{R} = \overline{\mathbb{Q}_{odd}}$. Even though $\mathfrak{F}_{\varphi} \cap \mathfrak{F}_{\psi} = \varnothing$, closing both folia in the weak*-topology on $\mathcal{A}^*$ yields the same collection of abstract states: ${}^{w^*}\overline{\mathfrak{F}}_{\varphi} = {}^{w^*}\overline{\mathfrak{F}}_{\psi}$. It would be a stretch to say, however, that $\mathbb{Q}_{even}$ and $\mathbb{Q}_{odd}$ are incommensurable theories of numbers. The more sensible thing to say is that there is a property, for example the property of being even or prime, that partitions the rational numbers $\mathbb{Q}$ into two disjoint sets. In a similar way, two disjoint factor representations will not have any folia states in common, but, according to converse of theorem 4.8.2, there will be an element $Z \in \mathfrak{Z}(\mathcal{A}^{**})$ that will distinguish them $\tilde{\varphi}(Z) \neq \tilde{\psi}(Z)$. It might be that this property is temperature, chemical potential, or something else. Notice that it is the larger sets of numbers such as $\mathbb{Q}$ and $\mathbb{R}$ that allow us to see how specific subsets of numbers such as $\mathbb{Q}_{even}$ and $\mathbb{Q}_{odd}$ differ. There are certain situations where we may only be interested in $\mathbb{Q}_{even}$ or $\mathbb{Q}_{odd}$, but we would have a severely restricted account of numbers if we did not realize how these sets are related to $\mathbb{Q}$ and $\mathbb{R}$. Thus, while $\pi_{\varphi}(A)''$ or $\pi_{\psi}(A)''$ might be useful for specific physical situations it is also crucial to understand that they are W*-representations of $\mathcal{A}^{**}$ and it is with reference to elements in $Z \in \mathfrak{Z}(\mathcal{A}^{**})$ that different UIRs can be distinguished. $\pi_{\varphi}(A)''$ or $\pi_{\psi}(A)''$ are more like different models of a physical theory $\mathcal{A}^{**}$ with, say different temperature values, than they are incommensurable theories.

## 5.11 CONCLUSIONS

While I have emphasized the differences between myself and Ruetsche, we do agree about many issues. We agree that UIRs have a significant role to play in characterizing the physical content of AQFT and AQSM. We agree that algebraic imperialism and the Hilbert space conservative do not succeed at capturing all of the physical content of AQFT and AQSM. We agree that to model specific physical situations only a subset of abstract states is needed. However, we disagree on where to locate the rest of the physical content not captured by algebraic imperialist and the Hilbert space conservative. Though Ruetsche hopes to distinguish her Hilbert space pluralist approach from Hilbert space universalism, it is not clear that this approach is successful at avoiding the mathematical surplus structure that she criticizes the Hilbert space universalist of having. If we want to capture the physical differences between UIRs in AQFT and AQSM, the observables in $\mathfrak{Z}\left(\mathcal{A}^{**}\right)$ are the key. This is the lesson of UIRs in both canonical QFT and AQFT: the original mathematical structures (the Fock representation, or *a*-representation, and the C*-algebra $\mathcal{A}$, respectively) are not sufficient to capture all of the physical content of QFT. Larger structures that contain all of the particular representations, such as the infinite tensor product of all Fock spaces and $\mathcal{A}^{**}$, contain the additional physical content that UIRs are pointing towards. By working with these larger spaces and algebras it is possible to compare different UIRs and construct new observables such that we truly "put unitarily inequivalent representations to work."

# Chapter Six

This dissertation has examined two extreme positions on UIRs: (1) UIRs are mathematical possibilities with no physical significance and (2) that different UIRs are incommensurable theories with radically different ontologies. But as the previous chapters have shown, the truth lies somewhere in between. Given certain assumptions, the Stone-von Neumann theorem proved that wave mechanics and matrix mechanics were not rival theories of quantum mechanics. Since they are unitarily equivalent, they are merely different formulations of essentially the same theory. It is not surprising then that the unitary *inequivalence* of two representations could be interpreted as incommensurable theories. The Unruh effect seems to reinforce the point that different UIRs have incompatible particle ontologies, or that particles may not exist. Some people view UIRs as arising only in the algebraic approach to QFT when odd situations such as accelerated and non-accelerated observers are considered. If UIRs can only be generated in very unusual situations in a particular formalism, then perhaps UIRs are not as philosophically interesting as they might appear on a first glance.

However, such conclusions are not supported by any serious study of UIRs in QFT. First, the number of UIRs is not reduced by using the canonical QFT framework instead of the algebraic approach or vice versa. In both frameworks, a continuum of UIRs can be generated using the coupling constant in canonical QFT or temperature in both AQSM and AQFT. Arguments for the preference of one representation over all other UIRs have been considered in both canonical QFT (Fock representation chauvinist) and AQFT (algebraic imperialist), however both positions have fatal flaws. The Fock representation chauvinist runs afoul of Haag's theorem in that she would only be able to model free quantum fields. The algebraic imperialist argument mainly depends on Haag and Kastler's notion of physical equivalence. This notion of physical equivalence can be applied to both C*-representations and W*-representations, but it only applies to a very limited number of W*-representations. However, most physical models use W*-representations – not C*-representations. For W*-representations, it is a simple matter to generate a continuum of UIRs which do not satisfy the mathematical requirements of Haag and Kastler's notion of physical equivalence. The existence of UIRs in both canonical QFT and AQFT show that the mathematical structures of the Fock representation and the C*-algebra respectively are insufficient to capture all of the physical content of QFT. UIRs can be thought of as pointers to additional physical content that cannot be captured in either mathematical structure. If these mathematical structures are not sufficient to capture this additional content, are there better options? Yes, there are larger mathematical structures that can capture this physical content, e.g., an infinite tensor product of Fock spaces in canonical QFT and the bidual $\mathcal{A}^{**}$ in the algebraic approach. Specific UIRs can be compared in terms of their expectation values of certain classical elements in $\mathcal{A}^{**}$. Different UIRs can be combined to build new observables such as temperature and chemical potential. In the algebraic approach, UIRs not only make an appearance in AQFT but also in AQSM. Using concepts from both AQFT and AQSM, the Unruh effect was shown to have not just two UIRs, but a continuum of UIRs of which no pair satisfies Haag and Kastler's notion of physical equivalence.

Given this radical non-uniqueness, is there a continuum of competing incommensurable quantum field theories each with a different ontology? No, in fact different UIRs share a fair amount of structure with each other. In the algebraic approach, they are both W*-representations of the same algebra: the bidual $\mathcal{A}^{**}$. Also, even though unitary equivalence no longer holds, an isomorphism may still hold between the representations. In canonical QFT, the proper unitary transformation becomes improper when a certain mathematical condition holds, but the improper transformation is still an isomorphism between the states and creation / annihilation operators of two UIRs. The worry that we suddenly lose our grip on QFT by having different UIRs with "radically" different ontologies becomes diffused by examining UIRs in canonical QFT where the mathematics is more straightforward and intuitive. It becomes clear how one mathematical condition causes unitary inequivalence and how the unitarily inequivalent Fock spaces are related to each other.

If UIRs are not different quantum field theories, how can we understand their relationship to each other? The most helpful picture is to think of them as different models of a physical theory. For example, in Newtonian mechanics a force can be modeled as a spring with a certain spring constant. Depending on the strength of the force in a particular situation, a different spring constant will be used. Similarly, in a particular situation a different coupling constant or temperature might be applicable in canonical QFT or AQFT respectively. In the case of the coupling constant, one would select the Fock space with the appropriate coupling constant from the continuum of UIRs in canonical QFT. For temperature, the W*-representation of a KMS state at that particular temperature would be selected from the continuum of UIRs with different temperatures in the algebraic approach. To think of Newtonian forces that differ only in the value of their spring constant as ontologically distinct forces is to misunderstand their relationship to each other.

The connection with models and UIRs can be extended and deepened. The dominant interpretation of AQFT has been operationalist, however I think a structural realist interpretation is more viable. The algebraic formulation of QFT has received surprisingly little attention in discussions of structural realism. One exception is Cao (2003), which establishes some connections between his version of structural realism and the algebraic approach. According to Cao, the algebra of observables is a set of empirically accessible relations describing structural features of hypothetical entities. The knowledge of the structural relationship of observables permits inferences about the physical reality of microscopic entities. It is not entirely clear what ontological inferences can be drawn from an algebra of observables. According to many advocates of AQFT, fields and particles are mere epiphenonmena and all physical content lies in the algebra of observables. Does AQFT possess a unique ontology or multiple distinct ontologies? To what extent can AQFT support a particle or field ontology? What would it mean to have an ontology based on an algebra of observables? My analysis of UIRs will have a crucial role in answering these questions. In light of this, I plan to develop an alternative version of structural realism more suited to the algebraic framework.

Another aspect of my dissertation that merits further investigation is the issue of dynamics in the algebraic formalism. Typically, the dynamics is described by an automorphism of the algebra, which is an isomorphism that maps all elements of an algebra to another element of the same algebra. In certain cases, the automorphism will be implementable by a unitary operator, in which case the automorphism is an inner automorphism. However, in the majority of cases the automorphism is not unitarily implementable, in which case the automorphism is an outer automorphism. There are even cases where the evolution cannot be described by *any* automorphism. I will develop a theory of time-evolution for W*-algebras to answer the following questions. Under what conditions will the dynamics be given by an inner or outer automorphism? Is there a more general type of mapping that can describe the evolution of these algebraic systems? If so, can such dynamics provide a possible solution to the measurement problem?

Finally, I want to explore some work that unites canonical QFT and AQFT. Chapter two examined how the particle content of canonical QFT is strongly tied to the existence and domain of a total number operator. There have been some attempts to define a total number operator in AQFT. What is particularly interesting is that alternative mathematical definitions have been proposed for a total number operator. There have also been attempts to define renormalized number operators. These different number operators are supposed to have some connection with different UIRs. I want to examine whether $\mathcal{A}^{**}$ can help clarify the relations between these different total number operators. This should have implications for the particle content in AQFT.

The search for a mathematically rigorous and conceptually clear formulation of QFT is an ongoing quest. There are many different frameworks for QFT each of which has certain advantages and disadvantages. The pluralism of frameworks (e.g., canonical QFT, AQFT, Wightman's axiomatic QFT, etc…) makes the job of a philosopher more difficult. Which framework has the best claim as representing the physical content of QFT? The matter is far from settled. Instead of arguing for the preference of one framework over the others, a specific topic of philosophical interest can be analyzed in different frameworks. Comparing the results in these frameworks can illuminate the deficiencies in these frameworks and may suggest alternative formulations that can overcome these limitations. This dissertation has taken this approach to analyzing QFT. Many questions remain. For example, can these alternative mathematical structures model interactions and the dynamics of systems in QFT? For now though, a more thorough understanding of UIRs in QFT and their philosophical significance must suffice.

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# Vita

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